MATH 304

Linear Algebra

Lecture 27:
Norms and inner products.

Norm

The notion of *norm* generalizes the notion of length of a vector in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\alpha : V \to \mathbb{R}$ is called a **norm** on V if it has the following properties:

(i) $\alpha(\mathbf{x}) \geq 0$, $\alpha(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\alpha(r\mathbf{x}) = |r|\,\alpha(\mathbf{x})$ for all $r \in \mathbb{R}$ (homogeneity)
(iii) $\alpha(\mathbf{x} + \mathbf{y}) \leq \alpha(\mathbf{x}) + \alpha(\mathbf{y})$ (triangle inequality)

Notation. The norm of a vector $\mathbf{x} \in V$ is usually denoted $\|\mathbf{x}\|$. Different norms on V are distinguished by subscripts, e.g., $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \dots, |x_n|)$.

Positivity and homogeneity are obvious.
The triangle inequality:

$$|x_i + y_i| \le |x_i| + |y_i| \le \max_j |x_j| + \max_j |y_j|$$
$$\implies \max_j |x_j + y_j| \le \max_j |x_j| + \max_j |y_j|$$

- $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.

Positivity and homogeneity are obvious.
The triangle inequality: $|x_i + y_i| \le |x_i| + |y_i|$

$$\implies \sum_j |x_j + y_j| \le \sum_j |x_j| + \sum_j |y_j|$$

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\big)^{1/p}$, $p > 0$.

Remark. $\|\mathbf{x}\|_2$ = Euclidean length of $\mathbf{x}$.

Theorem $\|\mathbf{x}\|_p$ is a norm on $\mathbb{R}^n$ for any $p \geq 1$.

Positivity and homogeneity are still obvious (and hold for any $p > 0$). The triangle inequality for $p \geq 1$ is known as the **Minkowski inequality**:

$$\big(|x_1 + y_1|^p + |x_2 + y_2|^p + \cdots + |x_n + y_n|^p\big)^{1/p} \leq$$
$$\leq \big(|x_1|^p + \cdots + |x_n|^p\big)^{1/p} + \big(|y_1|^p + \cdots + |y_n|^p\big)^{1/p}.$$

Normed vector space

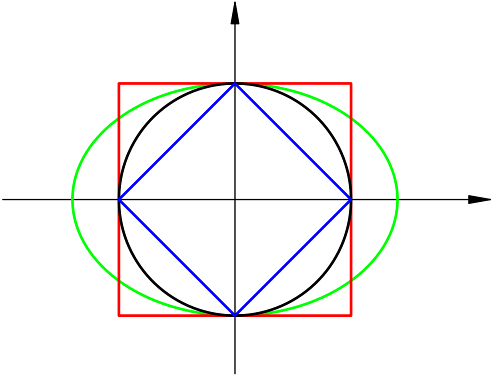
Definition. A **normed vector space** is a vector space endowed with a norm.

The norm defines a distance function on the normed vector space: $\operatorname{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Then we say that a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ *converges* to a vector $\mathbf{x}$ if $\operatorname{dist}(\mathbf{x}, \mathbf{x}_n) \to 0$ as $n \to \infty$.

Also, we say that a vector $\mathbf{x}$ is a good *approximation* of a vector $\mathbf{x}_0$ if $\operatorname{dist}(\mathbf{x}, \mathbf{x}_0)$ is small.

Unit circle: $\|\mathbf{x}\| = 1$



$\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$	black
$\|\mathbf{x}\| = \left(\frac{1}{2}x_1^2 + x_2^2\right)^{1/2}$	green
$\|\mathbf{x}\| = \lvert x_1\rvert + \lvert x_2\rvert$	blue
$\|\mathbf{x}\| = \max(\lvert x_1\rvert, \lvert x_2\rvert)$	red

Examples. $V = C[a,b]$, $f : [a,b] \to \mathbb{R}$.

- $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$.
- $\|f\|_1 = \int_a^b |f(x)|\,dx$.
- $\|f\|_p = \left(\int_a^b |f(x)|^p\,dx \right)^{1/p}$, $p > 0$.

Theorem $\|f\|_p$ is a norm on $C[a,b]$ for any $p \ge 1$.

Inner product

The notion of *inner product* generalizes the notion of dot product of vectors in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\beta : V \times V \to \mathbb{R}$, usually denoted $\beta(\mathbf{x}, \mathbf{y}) = \langle \mathbf{x}, \mathbf{y} \rangle$, is called an **inner product** on V if it is positive, symmetric, and bilinear. That is, if

(i) $\langle \mathbf{x}, \mathbf{x} \rangle \geq 0$, $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$ (symmetry)
(iii) $\langle r\mathbf{x}, \mathbf{y} \rangle = r\langle \mathbf{x}, \mathbf{y} \rangle$ (homogeneity)
(iv) $\langle \mathbf{x} + \mathbf{y}, \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{z} \rangle + \langle \mathbf{y}, \mathbf{z} \rangle$ (distributive law)

An **inner product space** is a vector space endowed with an inner product.

Examples. $V = \mathbb{R}^n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdots + x_ny_n$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = d_1x_1y_1 + d_2x_2y_2 + \cdots + d_nx_ny_n$,

where $d_1, d_2, \dots, d_n > 0$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = (D\mathbf{x}) \cdot (D\mathbf{y})$,

where D is an invertible $n \times n$ matrix.

Remarks. (a) Invertibility of D is necessary to show that $\langle \mathbf{x}, \mathbf{x} \rangle = 0 \implies \mathbf{x} = \mathbf{0}$.

(b) The second example is a particular case of the third one when $D = \mathrm{diag}(d_1^{1/2}, d_2^{1/2}, \dots, d_n^{1/2})$.

Counterexamples. $V = \mathbb{R}^2$.

- $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 - x_2 y_2$.

Let $\mathbf{v} = (1, 2)$, then $\langle \mathbf{v}, \mathbf{v} \rangle = 1^2 - 2^2 = -3$.
$\langle \mathbf{x}, \mathbf{y} \rangle$ is symmetric and bilinear, but not positive.

- $\langle \mathbf{x}, \mathbf{y} \rangle = 2x_1 y_1 + x_1 x_2 + 2x_2 y_2 + y_1 y_2$.

$\mathbf{v} = (1, 1)$, $\mathbf{w} = (1, 0) \implies \langle \mathbf{v}, \mathbf{w} \rangle = 3$, $\langle 2\mathbf{v}, \mathbf{w} \rangle = 8$.
$\langle \mathbf{x}, \mathbf{y} \rangle$ is positive and symmetric, but not bilinear.

- $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_1 y_2 - x_2 y_1 + x_2 y_2$.

$\mathbf{v} = (1, 1)$, $\mathbf{w} = (1, 0) \implies \langle \mathbf{v}, \mathbf{w} \rangle = 0$, $\langle \mathbf{w}, \mathbf{v} \rangle = 2$.
$\langle \mathbf{x}, \mathbf{y} \rangle$ is positive and bilinear, but not symmetric.

Problem. Find an inner product on $\mathbb{R}^2$ such that $\langle \mathbf{e}_1, \mathbf{e}_1 \rangle = 2$, $\langle \mathbf{e}_2, \mathbf{e}_2 \rangle = 3$, and $\langle \mathbf{e}_1, \mathbf{e}_2 \rangle = -1$, where $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$.

Let $\mathbf{x} = (x_1, x_2)$, $\mathbf{y} = (y_1, y_2) \in \mathbb{R}^2$.
Then $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$, $\mathbf{y} = y_1\mathbf{e}_1 + y_2\mathbf{e}_2$.
Using bilinearity, we obtain

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle x_1\mathbf{e}_1 + x_2\mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle$$

$$= x_1\langle \mathbf{e}_1, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle + x_2\langle \mathbf{e}_2, y_1\mathbf{e}_1 + y_2\mathbf{e}_2 \rangle$$

$$= x_1y_1\langle \mathbf{e}_1, \mathbf{e}_1 \rangle + x_1y_2\langle \mathbf{e}_1, \mathbf{e}_2 \rangle + x_2y_1\langle \mathbf{e}_2, \mathbf{e}_1 \rangle + x_2y_2\langle \mathbf{e}_2, \mathbf{e}_2 \rangle$$

$$= 2x_1y_1 - x_1y_2 - x_2y_1 + 3x_2y_2.$$

It remains to check that $\langle \mathbf{x}, \mathbf{x} \rangle > 0$ for $\mathbf{x} \neq \mathbf{0}$.
$\langle \mathbf{x}, \mathbf{x} \rangle = 2x_1^2 - 2x_1x_2 + 3x_2^2 = (x_1 - x_2)^2 + x_1^2 + 2x_2^2.$

Example. $V = \mathcal{M}_{m,n}(\mathbb{R})$, space of $m \times n$ matrices.

- $\langle A, B \rangle = \mathrm{trace}\,(AB^T)$.

If $A = (a_{ij})$ and $B = (b_{ij})$, then $\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} b_{ij}$.

Examples. $V = C[a, b]$.

- $\langle f, g \rangle = \int_a^b f(x)g(x)\,dx$.

- $\langle f, g \rangle = \int_a^b f(x)g(x)w(x)\,dx$,

where w is bounded, piecewise continuous, and $w > 0$ everywhere on $[a, b]$.

w is called the **weight** function.