

MATH 304  
Linear Algebra

**Lecture 38:**  
**Orthogonal polynomials.**

## Orthogonal polynomials

$\mathcal{P}$ : the vector space of all polynomials with real coefficients:  $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ .

Basis for  $\mathcal{P}$ :  $1, x, x^2, \dots, x^n, \dots$

Suppose that  $\mathcal{P}$  is endowed with an inner product.

*Definition.* **Orthogonal polynomials** (relative to the inner product) are polynomials  $p_0, p_1, p_2, \dots$  such that  $\deg p_n = n$  ( $p_0$  is a nonzero constant) and  $\langle p_n, p_m \rangle = 0$  for  $n \neq m$ .

Orthogonal polynomials can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis  $1, x, x^2, \dots$ :

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x),$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x),$$

.....

$$p_n(x) = x^n - \frac{\langle x^n, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \dots - \frac{\langle x^n, p_{n-1} \rangle}{\langle p_{n-1}, p_{n-1} \rangle} p_{n-1}(x),$$

.....

Then  $p_0, p_1, p_2, \dots$  are orthogonal polynomials.

**Theorem (a)** Orthogonal polynomials always exist.

**(b)** The orthogonal polynomial of a fixed degree is unique up to scaling.

**(c)** A polynomial  $p \neq 0$  is an orthogonal polynomial if and only if  $\langle p, q \rangle = 0$  for any polynomial  $q$  with  $\deg q < \deg p$ .

**(d)** A polynomial  $p \neq 0$  is an orthogonal polynomial if and only if  $\langle p, x^k \rangle = 0$  for any  $0 \leq k < \deg p$ .

*Proof of statement (b):* Suppose that  $P$  and  $R$  are two orthogonal polynomials of the same degree  $n$ . Then

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$  and

$R(x) = b_n x^n + b_{n-1} x^{n-1} + \cdots + b_1 x + b_0$ , where  $a_n, b_n \neq 0$ .

Consider a polynomial  $Q(x) = b_n P(x) - a_n R(x)$ . By construction,  $\deg Q < n$ . It follows from statement (c) that  $\langle P, Q \rangle = \langle R, Q \rangle = 0$ . Then

$$\langle Q, Q \rangle = \langle b_n P - a_n R, Q \rangle = b_n \langle P, Q \rangle - a_n \langle R, Q \rangle = 0,$$

which means that  $Q = 0$ . Thus  $R(x) = (a_n^{-1} b_n) P(x)$ .

*Example.*  $\langle p, q \rangle = \int_{-1}^1 p(x)q(x) dx.$

Note that  $\langle x^m, x^n \rangle = \int_{-1}^1 x^{m+n} dx = 0$  if  $m+n$  is odd. Hence  $p_{2k}(x)$  contains only even powers of  $x$  while  $p_{2k+1}(x)$  contains only odd powers of  $x$ .

$$p_0(x) = 1,$$

$$p_1(x) = x,$$

$$p_2(x) = x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = x^2 - \frac{1}{3},$$

$$p_3(x) = x^3 - \frac{\langle x^3, x \rangle}{\langle x, x \rangle} x = x^3 - \frac{3}{5}x.$$

$p_0, p_1, p_2, \dots$  are called the **Legendre polynomials**.

Instead of normalization, the orthogonal polynomials are subject to **standardization**.

The standardization for the Legendre polynomials is  $P_n(1) = 1$ . In particular,  $P_0(x) = 1$ ,  $P_1(x) = x$ ,  $P_2(x) = \frac{1}{2}(3x^2 - 1)$ ,  $P_3(x) = \frac{1}{2}(5x^3 - 3x)$ .

**Problem.** Find  $P_4(x)$ .

Let  $P_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ .

We know that  $P_4(1) = 1$  and  $\langle P_4, x^k \rangle = 0$  for  $0 \leq k \leq 3$ .

$$P_4(1) = a_4 + a_3 + a_2 + a_1 + a_0,$$

$$\langle P_4, 1 \rangle = \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0, \quad \langle P_4, x \rangle = \frac{2}{5}a_3 + \frac{2}{3}a_1,$$

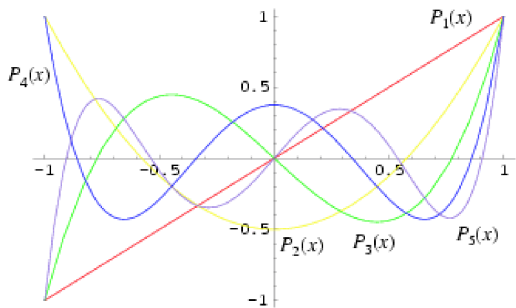
$$\langle P_4, x^2 \rangle = \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0, \quad \langle P_4, x^3 \rangle = \frac{2}{7}a_3 + \frac{2}{5}a_1.$$

$$\begin{cases} a_4 + a_3 + a_2 + a_1 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases}$$

$$\begin{cases} \frac{2}{5}a_3 + \frac{2}{3}a_1 = 0 \\ \frac{2}{7}a_3 + \frac{2}{5}a_1 = 0 \end{cases} \implies a_1 = a_3 = 0$$

$$\begin{cases} a_4 + a_2 + a_0 = 1 \\ \frac{2}{5}a_4 + \frac{2}{3}a_2 + 2a_0 = 0 \\ \frac{2}{7}a_4 + \frac{2}{5}a_2 + \frac{2}{3}a_0 = 0 \end{cases} \iff \begin{cases} a_4 = \frac{35}{8} \\ a_2 = -\frac{30}{8} \\ a_0 = \frac{3}{8} \end{cases}$$

Thus  $P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3)$ .



Legendre polynomials



**Problem.** Find a quadratic polynomial that is the best least squares fit to the function  $f(x) = |x|$  on the interval  $[-1, 1]$ .

The best least squares fit is a polynomial  $p(x)$  that minimizes the distance relative to the integral norm

$$\|f - p\| = \left( \int_{-1}^1 |f(x) - p(x)|^2 dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm  $\|f - p\|$  is minimal if  $p$  is the orthogonal projection of the function  $f$  on the subspace  $\mathcal{P}_3$  of polynomials of degree at most 2.

The Legendre polynomials  $P_0, P_1, P_2$  form an orthogonal basis for  $\mathcal{P}_3$ . Therefore

$$p(x) = \frac{\langle f, P_0 \rangle}{\langle P_0, P_0 \rangle} P_0(x) + \frac{\langle f, P_1 \rangle}{\langle P_1, P_1 \rangle} P_1(x) + \frac{\langle f, P_2 \rangle}{\langle P_2, P_2 \rangle} P_2(x).$$

$$\langle f, P_0 \rangle = \int_{-1}^1 |x| dx = 2 \int_0^1 x dx = 1,$$

$$\langle f, P_1 \rangle = \int_{-1}^1 |x| x dx = 0,$$

$$\langle f, P_2 \rangle = \int_{-1}^1 |x| \frac{3x^2 - 1}{2} dx = \int_0^1 x(3x^2 - 1) dx = \frac{1}{4},$$

$$\langle P_0, P_0 \rangle = \int_{-1}^1 dx = 2, \quad \langle P_2, P_2 \rangle = \int_{-1}^1 \left( \frac{3x^2 - 1}{2} \right)^2 dx = \frac{2}{5}.$$

$$\text{In general, } \langle P_n, P_n \rangle = \frac{2}{2n+1}.$$

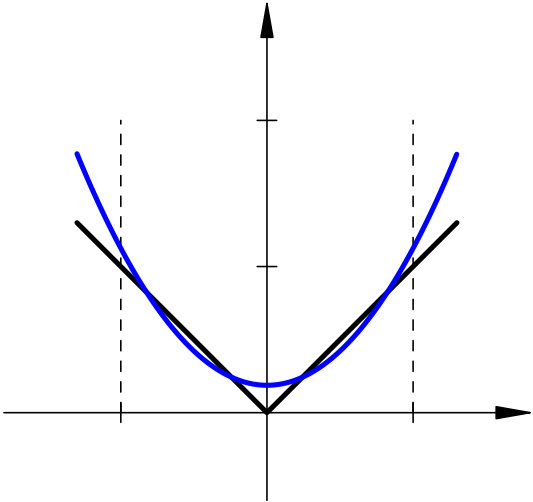
**Problem.** Find a quadratic polynomial that is the best least squares fit to the function  $f(x) = |x|$  on the interval  $[-1, 1]$ .

**Solution:** 
$$p(x) = \frac{1}{2}P_0(x) + \frac{5}{8}P_2(x)$$
$$= \frac{1}{2} + \frac{5}{16}(3x^2 - 1) = \frac{3}{16}(5x^2 + 1).$$

*Recurrent formula for the Legendre polynomials:*

$$(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x).$$

For example,  $4P_4(x) = 7xP_3(x) - 3P_2(x)$ .



*Definition.* **Chebyshev polynomials**  $T_0, T_1, T_2, \dots$  are orthogonal polynomials relative to the inner product

$$\langle p, q \rangle = \int_{-1}^1 \frac{p(x)q(x)}{\sqrt{1-x^2}} dx,$$

with the standardization  $T_n(1) = 1$ .

*Remark.* “T” is like in “Tschebyscheff”.

Change of variable in the integral:  $x = \cos \phi$ .

$$\begin{aligned} \langle p, q \rangle &= - \int_0^\pi \frac{p(\cos \phi) q(\cos \phi)}{\sqrt{1 - \cos^2 \phi}} \cos' \phi d\phi \\ &= \int_0^\pi p(\cos \phi) q(\cos \phi) d\phi. \end{aligned}$$

**Theorem.**  $T_n(\cos \phi) = \cos n\phi$ .

$$\begin{aligned}\langle T_n, T_m \rangle &= \int_0^\pi T_n(\cos \phi) T_m(\cos \phi) d\phi \\ &= \int_0^\pi \cos(n\phi) \cos(m\phi) d\phi = 0 \quad \text{if } n \neq m.\end{aligned}$$

Recurrent formula:  $T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$ .

$$T_0(x) = 1, \quad T_1(x) = x,$$

$$T_2(x) = 2x^2 - 1,$$

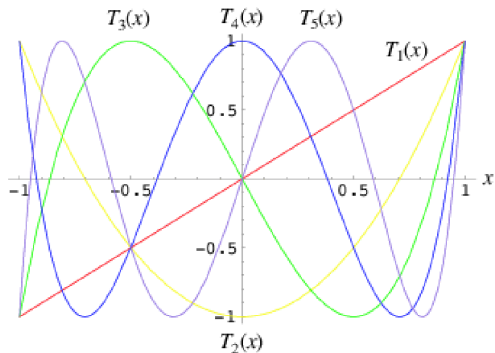
$$T_3(x) = 4x^3 - 3x,$$

$$T_4(x) = 8x^4 - 8x^2 + 1, \dots$$

That is,  $\cos 2\phi = 2 \cos^2 \phi - 1$ ,

$\cos 3\phi = 4 \cos^3 \phi - 3 \cos \phi$ ,

$\cos 4\phi = 8 \cos^4 \phi - 8 \cos^2 \phi + 1, \dots$



Chebyshev polynomials