MATH 304

Linear Algebra

Lecture 6:
Diagonal matrices.
Inverse matrix.

Matrices

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

Matrix algebra: linear operations

Addition: two matrices of the same dimensions can be added by adding their corresponding entries.

Scalar multiplication: to multiply a matrix A by a scalar r, one multiplies each entry of A by r.

Zero matrix O: all entries are zeros.

Negative: $-A$ is defined as $(-1)A$.

Subtraction: $A - B$ is defined as $A + (-B)$.

As far as the linear operations are concerned, the $m \times n$ matrices can be regarded as mn-dimensional vectors.

Matrix algebra: matrix multiplication

The product of matrices A and B is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times p$ matrix. The **product** AB is defined to be the $m \times p$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$ for all indices i, j.

That is, matrices are multiplied **row by column**.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array}\right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\implies \quad AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

Diagonal matrices

If $A = (a_{ij})$ is a square matrix, then the entries a_{ii} are called **diagonal entries**. A square matrix is called **diagonal** if all non-diagonal entries are zeros.

Example. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, denoted $\mathrm{diag}(7, 1, 2)$.

Let $A = \mathrm{diag}(s_1, s_2, \dots, s_n)$, $B = \mathrm{diag}(t_1, t_2, \dots, t_n)$.

Then $A + B = \mathrm{diag}(s_1 + t_1, s_2 + t_2, \dots, s_n + t_n)$,

$$rA = \mathrm{diag}(rs_1, rs_2, \dots, rs_n).$$

Example.

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Theorem Let $A = \operatorname{diag}(s_1, s_2, \ldots, s_n)$, $B = \operatorname{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \operatorname{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,

$$rA = \operatorname{diag}(rs_1, rs_2, \ldots, rs_n).$$

$$AB = \operatorname{diag}(s_1 t_1, s_2 t_2, \ldots, s_n t_n).$$

In particular, diagonal matrices always commute (i.e., $AB = BA$).

Example.

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 7a_{11} & 7a_{12} & 7a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix}$$

Theorem Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_m)$ and A be an $m \times n$ matrix. Then the matrix DA is obtained from A by multiplying the ith row by d_i for $i = 1, 2, \ldots, m$:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \implies DA = \begin{pmatrix} d_1\mathbf{v}_1 \\ d_2\mathbf{v}_2 \\ \vdots \\ d_m\mathbf{v}_m \end{pmatrix}$$

Example.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7a_{11} & a_{12} & 2a_{13} \\ 7a_{21} & a_{22} & 2a_{23} \\ 7a_{31} & a_{32} & 2a_{33} \end{pmatrix}$$

Theorem Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and A be an $m \times n$ matrix. Then the matrix AD is obtained from A by multiplying the ith column by d_i for $i = 1, 2, \ldots, n$:

$$A = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$$
$$\Longrightarrow \quad AD = (d_1\mathbf{w}_1, d_2\mathbf{w}_2, \ldots, d_n\mathbf{w}_n)$$

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted I_n or simply I.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We can **add**, **subtract**, and **multiply** elements of $\mathcal{M}_n(\mathbb{R})$. What about **division**?

Definition. Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

A non-invertible square matrix is called **singular**.

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = B$, $B^{-1} = A$, and $C^{-1} = C$.

Basic properties of inverse matrices

- If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1}, and $A = (A^{-1})^{-1}$.

- The inverse matrix (if it exists) is unique.

Moreover, if $AB = CA = I$ for some $n \times n$ matrices B and C, then $B = C = A^{-1}$.

Indeed, $B = IB = (CA)B = C(AB) = CI = C$.

- If $n \times n$ matrices A and B are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

- Similarly, $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.