MATH 304

Linear Algebra

Lecture 12:
Subspaces of vector spaces.

Abstract vector space

A *vector space* is a set V equipped with two operations, **addition** $V \times V \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \mathbf{y} \in V$ and **scalar multiplication** $\mathbb{R} \times V \ni (r, \mathbf{x}) \mapsto r\mathbf{x} \in V$, that have the following properties:

A1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for all $\mathbf{x}, \mathbf{y} \in V$;

A2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;

A3. there exists an element of V, called the *zero vector* and denoted $\mathbf{0}$, such that $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$;

A4. for any $\mathbf{x} \in V$ there exists an element of V, denoted $-\mathbf{x}$, such that $\mathbf{x} + (-\mathbf{x}) = (-\mathbf{x}) + \mathbf{x} = \mathbf{0}$;

A5. $r(\mathbf{x} + \mathbf{y}) = r\mathbf{x} + r\mathbf{y}$ for all $r \in \mathbb{R}$ and $\mathbf{x}, \mathbf{y} \in V$;

A6. $(r + s)\mathbf{x} = r\mathbf{x} + s\mathbf{x}$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$;

A7. $(rs)\mathbf{x} = r(s\mathbf{x})$ for all $r, s \in \mathbb{R}$ and $\mathbf{x} \in V$;

A8. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x} \in V$.

Additional properties of vector spaces

- The zero vector is unique.
- For any $\mathbf{x} \in V$, the negative $-\mathbf{x}$ is unique.
- $\mathbf{x} + \mathbf{y} = \mathbf{z} \iff \mathbf{x} = \mathbf{z} - \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- $\mathbf{x} + \mathbf{z} = \mathbf{y} + \mathbf{z} \iff \mathbf{x} = \mathbf{y}$ for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$.
- $0\mathbf{x} = \mathbf{0}$ for any $\mathbf{x} \in V$.
- $(-1)\mathbf{x} = -\mathbf{x}$ for any $\mathbf{x} \in V$.

Examples of vector spaces

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathcal{M}_{m,n}(\mathbb{R})$: $m\times n$ matrices with real entries
- $\mathbb{R}^\infty$: infinite sequences $(x_1, x_2, \dots)$, $x_i \in \mathbb{R}$
- $\{\mathbf{0}\}$: the trivial vector space
- $F(\mathbb{R})$: the set of all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$
- $C^1(\mathbb{R})$: all continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$
- $C^\infty(\mathbb{R})$: all smooth functions $f : \mathbb{R} \to \mathbb{R}$
- $\mathcal{P}$: all polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$

Subspaces of vector spaces

Definition. A vector space V_0 is a **subspace** of a vector space V if $V_0 \subset V$ and the linear operations on V_0 agree with the linear operations on V.

Examples.

- $F(\mathbb{R})$: all functions $f : \mathbb{R} \to \mathbb{R}$
- $C(\mathbb{R})$: all continuous functions $f : \mathbb{R} \to \mathbb{R}$

$C(\mathbb{R})$ is a subspace of $F(\mathbb{R})$.

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1x + \cdots + a_kx^k$
- $\mathcal{P}_n$: polynomials of degree less than n

$\mathcal{P}_n$ is a subspace of $\mathcal{P}$.

Subspaces of vector spaces

Counterexamples.

- $\mathbb{R}^n$: n-dimensional coordinate vectors
- $\mathbb{Q}^n$: vectors with rational coordinates

$\mathbb{Q}^n$ is not a subspace of $\mathbb{R}^n$.

$\sqrt{2}(1, 1, \ldots, 1) \notin \mathbb{Q}^n \implies \mathbb{Q}^n$ is not a vector space (scaling is not well defined).

- $\mathcal{P}$: polynomials $p(x) = a_0 + a_1 x + \cdots + a_n x^n$
- P_n^*: polynomials of degree n $(n > 0)$

P_n^* is not a subspace of $\mathcal{P}$.

$-x^n + (x^n + 1) = 1 \notin P_n^* \implies P_n^*$ is not a vector space (addition is not well defined).

If S is a subset of a vector space V then S inherits from V addition and scalar multiplication. However S need not be closed under these operations.

Proposition A subset S of a vector space V is a subspace of V if and only if S is **nonempty** and **closed under linear operations**, i.e.,

$$\mathbf{x}, \mathbf{y} \in S \implies \mathbf{x} + \mathbf{y} \in S,$$

$$\mathbf{x} \in S \implies r\mathbf{x} \in S \text{ for all } r \in \mathbb{R}.$$

Proof: "only if" is obvious.
"if": properties like associative, commutative, or distributive law hold for S because they hold for V. We only need to verify properties A3 and A4. Take any $\mathbf{x} \in S$ (note that S is nonempty). Then $\mathbf{0} = 0\mathbf{x} \in S$. Also, $-\mathbf{x} = (-1)\mathbf{x} \in S$. Thus $\mathbf{0}$ and $-\mathbf{x}$ in S are the same as in V.

Example. $V = \mathbb{R}^2$.

- The line $x - y = 0$ is a subspace of $\mathbb{R}^2$.

The line consists of all vectors of the form (t, t), $t \in \mathbb{R}$.

$(t, t) + (s, s) = (t + s, t + s) \implies$ closed under addition
$r(t, t) = (rt, rt) \implies$ closed under scaling

- The parabola $y = x^2$ is not a subspace of $\mathbb{R}^2$.

It is enough to find one explicit counterexample.

Counterexample 1: $(1, 1) + (-1, 1) = (0, 2)$.

$(1, 1)$ and $(-1, 1)$ lie on the parabola while $(0, 2)$ does not
$\implies$ not closed under addition

Counterexample 2: $2(1, 1) = (2, 2)$.

$(1, 1)$ lies on the parabola while $(2, 2)$ does not
$\implies$ not closed under scaling

Example. $V = \mathbb{R}^3$.

- The plane $z = 0$ is a subspace of $\mathbb{R}^3$.
- The plane $z = 1$ is not a subspace of $\mathbb{R}^3$.
- The line $t(1,1,0)$, $t \in \mathbb{R}$ is a subspace of $\mathbb{R}^3$ and a subspace of the plane $z = 0$.
- The line $(1,1,1) + t(1,-1,0)$, $t \in \mathbb{R}$ is not a subspace of $\mathbb{R}^3$ as it lies in the plane $x + y + z = 3$, which does not contain $\mathbf{0}$.
- In general, a straight line or a plane in $\mathbb{R}^3$ is a subspace if and only if it passes through the origin.

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Any solution $(x_1, x_2, \dots, x_n)$ is an element of $\mathbb{R}^n$.

Theorem The solution set of the system is a subspace of $\mathbb{R}^n$ if and only if all $b_i = 0$.

Theorem The solution set of a system of linear equations in n variables is a subspace of $\mathbb{R}^n$ if and only if all equations are homogeneous.

Proof: "only if": the zero vector $\mathbf{0} = (0, 0, \ldots, 0)$, which belongs to every subspace, is a solution only if all equations are homogeneous.

"if": a system of homogeneous linear equations is equivalent to a matrix equation $A\mathbf{x} = \mathbf{0}$, where A is the coefficient matrix of the system and all vectors are regarded as column vectors.

$A\mathbf{0} = \mathbf{0} \implies \mathbf{0}$ is a solution $\implies$ solution set is not empty.

If $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$ then $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$
$\implies$ solution set is closed under addition.

If $A\mathbf{x} = \mathbf{0}$ then $A(r\mathbf{x}) = r(A\mathbf{x}) = \mathbf{0}$
$\implies$ solution set is closed under scaling.

Thus the solution set is a subspace of $\mathbb{R}^n$.

Examples of subspaces of $\mathcal{M}_{2,2}(\mathbb{R})$*:* $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

- diagonal matrices: $b = c = 0$
- upper triangular matrices: $c = 0$
- lower triangular matrices: $b = 0$
- symmetric matrices ($A^T = A$): $b = c$
- anti-symmetric (or skew-symmetric) matrices ($A^T = -A$): $a = d = 0,\ c = -b$
- matrices with zero trace: $a + d = 0$ (trace = the sum of diagonal entries)
- matrices with zero determinant, $ad - bc = 0$,

do not form a subspace: $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.