MATH 304

Linear Algebra

Lecture 40:
Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.
- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.
- Matrix algebra. Inverse matrix.
- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.
- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product)
- Inner products and norms
- Orthogonal complement
- Least squares problems
- The Gram-Schmidt orthogonalization process
- Orthogonal polynomials

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Bases of eigenvectors, diagonalization
- Matrix exponentials
- Complex eigenvalues and eigenvectors
- Orthogonal matrices
- Rigid motions, rotations in space

Bases of eigenvectors

Let A be an $n \times n$ matrix with real entries.

- A has n distinct real eigenvalues $\implies$ a basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has complex eigenvalues $\implies$ no basis for $\mathbb{R}^n$ formed by eigenvectors of A
- A has n distinct complex eigenvalues $\implies$ a basis for $\mathbb{C}^n$ formed by eigenvectors of A
- A has multiple eigenvalues $\implies$ further information is needed
- an orthonormal basis for $\mathbb{R}^n$ formed by eigenvectors of A $\iff$ A is symmetric: $A^T = A$

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,

(b) a basis of eigenvectors for $\mathbb{C}^2$,

(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$ (a),(b),(c): yes

$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ (a),(b),(c): no

Problem. For each of the following 2×2 matrices determine whether it allows

(a) a basis of eigenvectors for $\mathbb{R}^2$,

(b) a basis of eigenvectors for $\mathbb{C}^2$,

(c) an orthonormal basis of eigenvectors for $\mathbb{R}^2$.

$$C = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$$ (a),(b): yes (c): no

$$D = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$ (b): yes (a),(c): no

Problem. Let V be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),$$
$$\begin{aligned} f_2'(x) &= (x \cos x)' = -x \sin x + \cos x \\ &= -f_1(x) + f_4(x), \end{aligned}$$
$$f_3'(x) = (\sin x)' = \cos x = f_4(x),$$
$$f_4'(x) = (\cos x)' = -\sin x = -f_3(x).$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A-\lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2+1) + (\lambda^2+1) = (\lambda^2+1)^2 = (\lambda-i)^2(\lambda+i)^2.$$

The roots are i and $-i$, both of multiplicity 2.

One can show that both eigenspaces of A are one-dimensional. The eigenspace for i is spanned by $(0, 0, i, 1)$ and the eigenspace for $-i$ is spanned by $(0, 0, -i, 1)$. It follows that the matrix A is not diagonalizable in $\mathbb{C}^4$.

There is also an indirect way to show that A is not diagonalizable in $\mathbb{C}^4$. Assume the contrary. Then $A = UPU^{-1}$, where U is an invertible matrix with complex entries and

$$P = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

(note that P should have the same characteristic polynomial as A). This would imply that $A^2 = UP^2U^{-1}$. But $P^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

Let us check if $A^2 = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, we have a contradiction. Thus the matrix A is not diagonalizable in $\mathbb{C}^4$.

Problem. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{2014} (L applied 2014 times).

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \quad \mathbf{v}_0 = (3/5, 0, -4/5).$$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -4/5 \\ y & z \end{vmatrix} \mathbf{e}_1 - \begin{vmatrix} 3/5 & -4/5 \\ x & z \end{vmatrix} \mathbf{e}_2 + \begin{vmatrix} 3/5 & 0 \\ x & y \end{vmatrix} \mathbf{e}_3$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3 = \left(\tfrac{4}{5}y, -\tfrac{4}{5}x - \tfrac{3}{5}z, \tfrac{3}{5}y\right).$$

In particular, $L(\mathbf{e}_1) = \left(0, -\frac{4}{5}, 0\right)$, $L(\mathbf{e}_2) = \left(\frac{4}{5}, 0, \frac{3}{5}\right)$, $L(\mathbf{e}_3) = \left(0, -\frac{3}{5}, 0\right)$.

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}$.

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Longrightarrow \; x + \tfrac{3}{4}z = y = 0 \; \Longrightarrow \; \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.
It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{2014} is B^{2014}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{2014} = UD^{2014}U^{-1}$. We have that $D^{2014} =$
$= \mathrm{diag}\big(0, i^{2014}, (-i)^{2014}\big) = \mathrm{diag}(0, -1, -1) = D^2$.
Hence

$$B^{2014} = UD^2U^{-1} = B^2 = \begin{pmatrix} -0.64 & 0 & -0.48 \\ 0 & -1 & 0 \\ -0.48 & 0 & -0.36 \end{pmatrix}.$$

Problem. Let R denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $R(\mathbf{e}_1) = \mathbf{e}_2$, $R(\mathbf{e}_2) = \mathbf{e}_3$, $R(\mathbf{e}_3) = \mathbf{e}_1$. Describe R in geometric terms.

Alternative solution: The operator R maps one orthonormal basis to an orthonormal basis (namely, the standard basis is mapped to itself). Therefore R is a rigid motion. According to the classification of linear isometries in $\mathbb{R}^3$, R is either a rotation about an axis, or a reflection in a plane, or the composition of two.

Note that $R^3(\mathbf{e}_1) = R(R(R(\mathbf{e}_1))) = R(R(\mathbf{e}_2)) = R(\mathbf{e}_3) = \mathbf{e}_1$. Likewise, $R^3(\mathbf{e}_2) = \mathbf{e}_2$ and $R^3(\mathbf{e}_3) = \mathbf{e}_3$. Since R^3 is linear, it is the identity map. Now it follows that R preserves orientation and so is a rotation. Let ϕ be the angle of rotation, $0 \leq \phi \leq \pi$. Then R^3 is a rotation by 3ϕ. Since R^3 is the identity, we obtain that $3\phi = 2\pi$. The axis of rotation is the line spanned by $(1,1,1)$ since $R(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) =$
$= R(\mathbf{e}_1) + R(\mathbf{e}_2) + R(\mathbf{e}_3) = \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_1$.