Quiz 1: Solution

Problem. Find a matrix exponential $\exp(A)$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Solution: $\exp(A) = \dfrac{1}{2}\begin{pmatrix} e^2+1 & e^2-1 \\ e^2-1 & e^2+1 \end{pmatrix}$.

The diagonalization of the matrix A is $A = UDU^{-1}$, where

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Namely, 0 and 2 are the eigenvalues of A while $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are the associated eigenvectors. Then

$$e^A = Ue^DU^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^0 & 0 \\ 0 & e^2 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Alternative solution: One can check that $A^2 = 2A$. Then $A^3 = A^2A = (2A)A = 2A^2 = 2(2A) = 2^2A$, $A^4 = A^3A = (2^2A)A = 2^2A^2 = 2^2(2A) = 2^3A$, and so on. In general, $A^n = 2^{n-1}A$ for any integer $n \geq 1$. Therefore

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots = I + cA,$$

where

$$c = 1 + \frac{2}{2!} + \cdots + \frac{2^{n-1}}{n!} + \cdots$$

We know that

$$e^2 = 1 + 2 + \frac{2^2}{2!} + \cdots + \frac{2^n}{n!} + \cdots$$

It follows that $c = (e^2-1)/2$. Thus

$$\exp(A) = I + \frac{e^2-1}{2}A.$$