MATH 304–503 December 3, 2014

Quiz 1: Solution

Problem. Find a matrix exponential $\exp(A)$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Solution: $\exp(A) = \dfrac{1}{2}\begin{pmatrix} e+e^{-1} & e-e^{-1} \\ e-e^{-1} & e+e^{-1} \end{pmatrix}$.

The diagonalization of the matrix A is $A = UDU^{-1}$, where

$$D = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Namely, -1 and 1 are the eigenvalues of A while $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are the associated eigenvectors. Then

$$e^A = Ue^DU^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} e^{-1} & 0 \\ 0 & e \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}.$$

Alternative solution: One can check that $A^2 = I$. It follows that $A^n = I$ for any even integer $n > 0$ and $A^n = A$ for any odd integer $n > 0$. Therefore

$$\exp(A) = I + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \cdots = c_0 I + c_1 A,$$

where

$$c_0 = 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots + \frac{1}{(2k)!} + \cdots,$$

$$c_1 = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots + \frac{1}{(2k+1)!} + \cdots$$

We know that

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!} + \cdots,$$

$$e^{-1} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!} + \cdots$$

It follows that $c_0 = (e + e^{-1})/2$ and $c_1 = (e - e^{-1})/2$. Thus

$$\exp(A) = \frac{e+e^{-1}}{2}\, I + \frac{e-e^{-1}}{2}\, A.$$