MATH 304–510 Spring 2017

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1. Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Problem 2. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.
(ii) Find the inverse matrix A^{-1}.

Problem 3. Consider a linear transformation $F : \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$F(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_3 + x_5,\ 2x_1 - x_2 + x_4).$$

Find a basis for the kernel of F, then extend it to a basis for $\mathbb{R}^5$.

Problem 4. Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
(iii) Find the matrix of the operator L relative to the standard basis.

Problem 5. Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

Problem 6. Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

(i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V.
(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

Problem 7. Suppose M is an $n \times n$ matrix. Prove that there exists a nonzero polynomial $p(x)$ of degree at most n^2 such that $p(M) = O$.

Problem 8. Consider a linear operator $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(\mathbf{x}) = C\mathbf{x}, \quad \text{where} \quad C = \frac{1}{9}\begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

(i) Explain why K is a rigid motion and, specifically, a rotation about an axis.
(ii) Find the axis of rotation.
(iii) Find the angle of rotation.

Problem 9. Let P be a square matrix. Assuming P is diagonalizable, prove that $\det(\exp P) = e^{\mathrm{trace}(P)}$.