MATH 304–510 Spring 2017

Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Problem 1. Find a quadratic polynomial $p(x)$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Let $p(x) = a + bx + cx^2$. Then $p(-1) = a - b + c$, $p(1) = a + b + c$, and $p(3) = a + 3b + 9c$. Also, $p'(x) = b + 2cx$ so that $p'(2) = b + 4c$. The coefficients a, b, and c are to be chosen so that

$$\begin{cases} a - b + c = 6, \\ a + 3b + 9c = 6, \\ b + 4c = a + b + c \end{cases} \iff \begin{cases} a - b + c = 6, \\ a + 3b + 9c = 6, \\ a - 3c = 0. \end{cases}$$

This is a system of linear equations. To solve it, we convert the augmented matrix to reduced row echelon form using elementary row operations:

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 6 \\ 1 & 3 & 9 & 6 \\ 1 & 0 & -3 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 1 & -1 & 1 & 6 \\ 1 & 3 & 9 & 6 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 1 & 3 & 9 & 6 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 0 & 3 & 12 & 6 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 24 & 24 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & -1 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & 1 & -4 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -3 & 0 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

We obtain that the system has a unique solution: $a = 3$, $b = -2$, and $c = 1$. Thus $p(x) = x^2 - 2x + 3$.

Problem 2. Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

The determinant of A is easily evaluated using column expansions:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 2 & 3 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = 1.$$

Another way to evaluate $\det A$ is to convert the matrix A into the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of A.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the first row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right).$$

Interchange the second row with the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array}\right).$$

Subtract the second row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array}\right).$$

Interchange the third row with the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array}\right).$$

Add the fourth row to the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array}\right).$$

Subtract the second row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & 2 & 0 & 1 & -1 \end{array}\right).$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 \\ 2 & 3 & 0 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 0 & 0 & -1 \\ -2 & 0 & 0 & 1 \\ 1 & 1 & 1 & -1 \\ 2 & 0 & 1 & -1 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of A. We have transformed A into the identity matrix using elementary row operations. These included two row exchanges and no row multiplications. It follows that $\det A = \det I = 1$.

Problem 3. Consider a linear transformation $F : \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$F(x_1, x_2, x_3, x_4, x_5) = (x_1 + x_3 + x_5,\ 2x_1 - x_2 + x_4).$$

Find a basis for the kernel of F, then extend it to a basis for $\mathbb{R}^5$.

The kernel of F consists of all vectors $\mathbf{x} \in \mathbb{R}^5$ such that $F(\mathbf{x}) = \mathbf{0}$. This is the solution set of the following systems of linear equations:

$$\begin{cases} x_1 + x_3 + x_5 = 0 \\ 2x_1 - x_2 + x_4 = 0 \end{cases} \iff \begin{cases} x_1 + x_3 + x_5 = 0 \\ -x_2 - 2x_3 + x_4 - 2x_5 = 0 \end{cases}$$

$$\iff \begin{cases} x_1 + x_3 + x_5 = 0 \\ x_2 + 2x_3 - x_4 + 2x_5 = 0 \end{cases} \iff \begin{cases} x_1 = -x_3 - x_5 \\ x_2 = -2x_3 + x_4 - 2x_5 \end{cases}$$

The general solution of the system is

$$\mathbf{x} = (-t_1 - t_3, -2t_1 + t_2 - 2t_3, t_1, t_2, t_3) = t_1(-1, -2, 1, 0, 0) + t_2(0, 1, 0, 1, 0) + t_3(-1, -2, 0, 0, 1),$$

where t_1, t_2, t_3 are arbitrary real numbers. We obtain that the kernel of F is spanned by vectors $\mathbf{v}_1 = (-1, -2, 1, 0, 0)$, $\mathbf{v}_2 = (0, 1, 0, 1, 0)$, and $\mathbf{v}_3 = (-1, -2, 0, 0, 1)$. The last three coordinates of these vectors form the standard basis for $\mathbb{R}^3$. It follows that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. Hence they form a basis for the kernel.

To extend the basis for the kernel of F to a basis for $\mathbb{R}^5$, we need two more vectors. We can use two vectors from the standard basis. For example, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_1, \mathbf{e}_2$ form a basis for $\mathbb{R}^5$. To verify this, we show that a 5×5 matrix with these vectors as columns has a nonzero determinant:

$$\begin{vmatrix} -1 & 0 & -1 & 1 & 0 \\ -2 & 1 & -2 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 0 & -1 \\ 0 & 1 & -2 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 1.$$

Problem 4. Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

Let U be a 3×3 matrix such that its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the determinant of U, we subtract the second row from the first one and then expand by the first row:

$$\det U = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Since $\det U \neq 0$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. It follows that they form a basis for $\mathbb{R}^3$.

(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let A denote the matrix of L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By definition, the columns of A are coordinates of vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since $L(\mathbf{v}_1) = \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$, $L(\mathbf{v}_2) = \mathbf{v}_3 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$, we obtain

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(iii) Find the matrix of the operator L relative to the standard basis.

Let S denote the matrix of L relative to the standard basis for $\mathbb{R}^3$. We have $S = UAU^{-1}$, where A is the matrix of L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (already found) and U is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are consecutive columns of U):

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the inverse U^{-1}, we merge the matrix U with the identity matrix I into one 3×6 matrix and apply row reduction to convert the left half U of this matrix into I. Simultaneously, the right half I will be converted into U^{-1}:

$$(U|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = (I|U^{-1}).$$

Thus

$$S = UAU^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Alternative solution: Let S denote the matrix of L relative to the standard basis $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$. By definition, the columns of S are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$. It is easy to observe that $\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$, $\mathbf{e}_3 = \mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{e}_2 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Therefore

$$\begin{aligned} L(\mathbf{e}_1) &= L(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = -L(\mathbf{v}_1) + L(\mathbf{v}_2) + L(\mathbf{v}_3) = -\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_1 = (1,0,2), \\ L(\mathbf{e}_2) &= L(\mathbf{v}_1 - \mathbf{v}_3) = L(\mathbf{v}_1) - L(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1 = (0,0,-1), \\ L(\mathbf{e}_3) &= L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{v}_2 - \mathbf{v}_3 = (0,1,-1). \end{aligned}$$

Thus

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

Problem 5. Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 2$$

$$= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1 - \lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3 - \lambda).$$

Hence the matrix B has two eigenvalues: 0 and 3.

(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently, $\mathbf{x} = t(-1, 1, 0) + s(-1, 0, 1)$. Hence the eigenspace of B associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$.

Now consider the case $\lambda = 3$. We obtain that

$$(B - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = y = z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B. They are linearly independent since the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.

It is easy to check that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. To transform the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into an orthogonal one, we only need to orthogonalize the pair $\mathbf{v}_1, \mathbf{v}_2$. Using the Gram-Schmidt process, we replace the vector $\mathbf{v}_2$ by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = (-1/2, -1/2, 1).$$

Now $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$ is an orthogonal basis for $\mathbb{R}^3$. Since $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, it is also an eigenvector of B associated with the eigenvalue 0.

Finally, vectors $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$, $\mathbf{w}_2 = \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$, and $\mathbf{w}_3 = \dfrac{\mathbf{v}_3}{\|\mathbf{v}_3\|}$ form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B. We get that $\|\mathbf{v}_1\| = \sqrt{2}$, $\|\mathbf{u}\| = \sqrt{3/2}$, and $\|\mathbf{v}_3\| = \sqrt{3}$. Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1, -1, 2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B associated with eigenvalues 0, 0, and 3, respectively. Since these vectors form a basis for $\mathbb{R}^3$, it follows that $B = UDU^{-1}$, where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Here U is the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$) while D is the matrix of the linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$, $L(\mathbf{x}) = B\mathbf{x}$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Problem 6. Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.

(i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V.

(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

The vector $\mathbf{y}$ is uniquely represented as $\mathbf{y} = \mathbf{p} + \mathbf{o}$, where $\mathbf{p} \in V$ and $\mathbf{o}$ is orthogonal to V, that is, $\mathbf{o} \in V^\perp$. The vector $\mathbf{p}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace V. Since $(V^\perp)^\perp = V$, the vector $\mathbf{o}$ is the orthogonal projection of $\mathbf{y}$ onto the subspace $V^\perp$. It follows that the distance from the point $\mathbf{y}$ to V equals $\|\mathbf{o}\|$ while the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{p}\|$.

The orthogonal projection $\mathbf{p}$ of the vector $\mathbf{y}$ onto the subspace V is easily computed when we have an orthogonal basis for V. To get such a basis, we apply the Gram-Schmidt orthogonalization process to the basis $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$:

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 0, 0), \qquad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (2, 0, -1, 1) - \frac{2}{2}(1, 1, 0, 0) = (1, -1, -1, 1),$$

$$\mathbf{v}_3 = \mathbf{x}_3 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 - \frac{\mathbf{x}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = (0, 1, 1, 0) - \frac{1}{2}(1, 1, 0, 0) - \frac{-2}{4}(1, -1, -1, 1) = (0, 0, 1/2, 1/2).$$

Now that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthogonal basis for V we obtain

$$\mathbf{p} = \frac{\mathbf{y} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{y} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{y} \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3}\mathbf{v}_3 =$$

$$= \frac{0}{2}(1, 1, 0, 0) + \frac{4}{4}(1, -1, -1, 1) + \frac{2}{1/2}(0, 0, 1/2, 1/2) = (1, -1, 1, 3).$$

Consequently, $\mathbf{o} = \mathbf{y} - \mathbf{p} = (0, 0, 0, 4) - (1, -1, 1, 3) = (-1, 1, -1, 1)$. Thus the distance from $\mathbf{y}$ to the subspace V equals $\|\mathbf{o}\| = 2$ and the distance from $\mathbf{y}$ to $V^\perp$ equals $\|\mathbf{p}\| = \sqrt{12} = 2\sqrt{3}$.

Problem 7. Suppose M is an $n \times n$ matrix. Prove that there exists a nonzero polynomial $p(x)$ of degree at most n^2 such that $p(M) = O$.

All $n \times n$ matrices form a vector space of dimension n^2. It follows that any $n^2 + 1$ matrices in this space are linearly dependent. In particular, the matrices $I, M, M^2, \dots, M^{n^2}$ are linearly dependent. That is, $a_0 I + a_1 M + a_2 M^2 + \cdots + a_{n^2} M^{n^2} = O$ for some scalars $a_0, a_1, \dots, a_{n^2}$ not all equal to 0. Then $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n^2} x^{n^2}$ is the required polynomial.

Remark. According to the Cayley-Hamilton theorem, the characteristic polynomial of the matrix M (which has degree n) can be chosen as $p(x)$.

Problem 8. Consider a linear operator $K : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$K(\mathbf{x}) = C\mathbf{x}, \quad \text{where} \quad C = \frac{1}{9}\begin{pmatrix} -4 & 7 & 4 \\ 1 & -4 & 8 \\ 8 & 4 & 1 \end{pmatrix}.$$

(i) Explain why K is a rigid motion and, specifically, a rotation about an axis.

The matrix C is orthogonal, $CC^T = C^T C = I$. Therefore K is a rigid motion. According to the classification of linear isometries in $\mathbb{R}^3$, K is either a rotation about an axis, or a reflection in a plane, or the composition of two. Since $\det C = 1 > 0$, the transformation K preserves orientation. Hence K is a rotation.

(ii) Find the axis of rotation.

The axis of rotation is the set of points fixed by the operator K. Hence a point $\mathbf{x} \in \mathbb{R}^3$ lies on the axis if and only if $K(\mathbf{x}) = \mathbf{x}$ or, equivalently, $(C - I)\mathbf{x} = \mathbf{0}$. To solve this vector equation, we convert the matrix $C - I$ to reduced row echelon form:

$$C - I = \frac{1}{9}\begin{pmatrix} -13 & 7 & 4 \\ 1 & -13 & 8 \\ 8 & 4 & -8 \end{pmatrix} \to \frac{1}{9}\begin{pmatrix} 3 & 15 & -12 \\ 1 & -13 & 8 \\ 8 & 4 & -8 \end{pmatrix} \to \frac{1}{9}\begin{pmatrix} 3 & 15 & -12 \\ 9 & -9 & 0 \\ 8 & 4 & -8 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 5 & -4 \\ 1 & -1 & 0 \\ 2 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 1 & 5 & -4 \\ 2 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & -4 \\ 2 & 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 6 & -4 \\ 0 & 3 & -2 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 3 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -2/3 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here is the list of performed operations: add 2 times the third row to the first row, add the third row to the second row, multiply the first row by 3 and the third row by 9/4, interchange the first row with the second row, subtract the first row from the second row, subtract 2 times the first row from the third row, subtract 2 times the third row from the second row, interchange the second row with the third row, multiply the second row by 1/3, and add the second row to the first row.

Therefore a point $\mathbf{x} = (x, y, z)$ lies on the axis if and only if $x - \frac{2}{3}z = y - \frac{2}{3}z = 0$. The general solution of the system is $x = y = \frac{2}{3}t$, $z = t$, where $t \in \mathbb{R}$. Thus the axis of rotation is the line spanned by the vector $(2, 2, 3)$.

(iii) Find the angle of rotation.

Since K is a rotation about an axis, the matrix C is similar to the matrix

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

where ϕ is the angle of rotation. Similar matrices have the same trace (the sum of diagonal entries) as similar matrices have the same characteristic polynomial and the trace is one of its coefficients. Since $\text{trace}(C) = -7/9$ and $\text{trace}(E) = 1 + 2\cos\phi$, we obtain $1 + 2\cos\phi = -7/9$. Then $\cos\phi = -8/9$ so that $\phi = \arccos(-8/9)$.

Problem 9. Let P be a square matrix. Assuming P is diagonalizable, prove that $\det(\exp P) = e^{\text{trace}(P)}$.

First consider the case when P is diagonal, $P = \text{diag}(a_1, a_2, \dots, a_n)$. Then the matrix exponential e^P is also diagonal, namely, $e^P = \text{diag}(e^{a_1}, e^{a_2}, \dots, e^{a_n})$. The determinant of a diagonal matrix equals the product of its diagonal entries. Hence $\det(e^P) = e^{a_1}e^{a_2}\dots e^{a_n} = e^{a_1+a_2+\cdots+a_n} = e^{\text{trace}(P)}$.

Now assume that the matrix P is diagonalizable. Then it is similar to a diagonal matrix, that is, $P = UQU^{-1}$, where Q is diagonal. It follows that $e^P = Ue^QU^{-1}$. In particular, e^P is similar to e^Q. By the above, $\det(e^Q) = e^{\text{trace}(Q)}$. Since similar matrices have the same determinant and trace, we obtain $\det(e^P) = \det(e^Q)$ and $\text{trace}(P) = \text{trace}(Q)$. Hence $\det(e^P) = e^{\text{trace}(P)}$.