

MATH 304
Linear Algebra

Lecture 2:
Gaussian elimination.

System of linear equations

$$\left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right.$$

Here x_1, x_2, \dots, x_n are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

A system of linear equations can have **one** solution, **infinitely many** solutions, or **no** solution at all.

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

$$x - y = -2 \implies x = y - 2$$

$$2x + 3y = 6 \implies 2(y - 2) + 3y = 6$$

After the elimination is completed, the system is solved by *back substitution*.

$$y = 2 \implies x = y - 2 = 0$$

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

After elimination:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

After back substitution:

$$\begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solve the 1st equation for x :

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Eliminate x from the 3rd equation:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -(-y + 2z + 1) + 4y - 3z = 14 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Solve the 2nd equation for y :

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5y - 5z = 15 \end{cases}$$

Eliminate y from the 3rd equation:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5(z + 3) - 5z = 15 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 0 = 0 \end{cases}$$

The elimination is completed.

The last equation is actually $0z = 0$. Hence z is a *free variable*, i.e., it can be assigned an arbitrary value. Then y and x are found by back substitution.

$$z = t, \text{ a parameter;}$$

$$y = z + 3 = t + 3;$$

$$x = -y + 2z + 1 = -(t + 3) + 2t + 1 = t - 2.$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

General solution:

$$(x, y, z) = (t - 2, t + 3, t), \quad t \in \mathbb{R}.$$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1)$.

The set of all solutions is a straight line in \mathbb{R}^3 passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1)$.

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solve the 1st equation for x :

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Eliminate x from the 3rd equation:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ -(-y + 2z + 1) + 4y - 3z = 1 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y - z = 3 \\ 5y - 5z = 2 \end{cases}$$

Solve the 2nd equation for y :

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5y - 5z = 2 \end{cases}$$

Eliminate y from the 3rd equation:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 5(z + 3) - 5z = 2 \end{cases}$$

Simplify:

$$\begin{cases} x = -y + 2z + 1 \\ y = z + 3 \\ 15 = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Gaussian elimination

Gaussian elimination is a modification of the elimination method that allows only so-called *elementary operations*.

Elementary operations for systems of linear equations:

- (1) to multiply an equation by a nonzero scalar;
- (2) to add an equation multiplied by a scalar to another equation;
- (3) to interchange two equations.

Theorem (i) Applying elementary operations to a system of linear equations does not change the solution set of the system. **(ii)** Any elementary operation can be undone by another elementary operation.

Operation 1: multiply the i th equation by $r \neq 0$.

$$\begin{aligned} & \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right. \\ \implies & \left\{ \begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \dots\dots\dots \\ (ra_{i1})x_1 + (ra_{i2})x_2 + \cdots + (ra_{in})x_n = rb_i \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array} \right. \end{aligned}$$

To undo the operation, multiply the i th equation by r^{-1} .

Operation 2: add r times the i th equation to the j th equation.

$$\left\{ \begin{array}{c} \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \\ \dots\dots\dots \end{array} \right. \implies$$

$$\left\{ \begin{array}{c} \dots\dots\dots \\ a_{i1}x_1 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \\ (a_{j1} + ra_{i1})x_1 + \dots + (a_{jn} + ra_{in})x_n = b_j + rb_i \\ \dots\dots\dots \end{array} \right.$$

To undo the operation, add $-r$ times the i th equation to the j th equation.

Operation 3: interchange the i th and j th equations.

$$\begin{aligned} & \left\{ \begin{array}{c} \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \\ \dots\dots\dots \end{array} \right. \\ \Rightarrow & \left\{ \begin{array}{c} \dots\dots\dots \\ a_{j1}x_1 + a_{j2}x_2 + \dots + a_{jn}x_n = b_j \\ \dots\dots\dots \\ a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n = b_i \\ \dots\dots\dots \end{array} \right. \end{aligned}$$

To undo the operation, apply it once more.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Add -2 times the 1st equation to the 2nd equation:

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ x + y + z & = 6 \end{cases} \quad \boxed{R_2 := R_2 - 2 * R_1}$$

Add -1 times the 1st equation to the 3rd equation:

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ 2y + z & = 4 \end{cases}$$

Add -2 times the 2nd equation to the 3rd equation:

$$\begin{cases} x - y & = & 2 \\ & y - z & = & -1 \\ & & 3z & = & 6 \end{cases}$$

The elimination is completed, and we can solve the system by back substitution. However we can as well proceed with elementary operations.

Multiply the 3rd equation by $1/3$:

$$\begin{cases} x - y & = & 2 \\ & y - z & = & -1 \\ & & z & = & 2 \end{cases}$$

Add the 3rd equation to the 2nd equation:

$$\begin{cases} x - y & = 2 \\ y & = 1 \\ z & = 2 \end{cases}$$

Add the 2nd equation to the 1st equation:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z & = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$