MATH 304

Linear Algebra

**Lecture 3:
Row echelon form.
Gauss-Jordan reduction.**

Gaussian elimination

Solution of a system of linear equations splits into two parts:
(A) elimination and **(B)** back substitution. Both parts can be
done by applying a finite number of **elementary operations**:

(1) to multiply an equation by a nonzero scalar;
(2) to add an equation multiplied by a scalar to another
equation;
(3) to interchange two equations.

Example.

$$\begin{cases} x & - & y & & & = & 2 \\ 2x & - & y & - & z & = & 3 \\ x & + & y & + & z & = & 6 \end{cases} \rightarrow \begin{cases} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & 2y & + & z & = & 4 \end{cases}$$

$$\rightarrow \begin{cases} x & - & y & & & = & 2 \\ & & y & - & z & = & -1 \\ & & & & 3z & = & 6 \end{cases} \rightarrow \begin{cases} x & & & & & = & 3 \\ & & y & & & = & 1 \\ & & & & z & = & 2 \end{cases}$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 5y - 5z = 15 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases}$$

Add -1 times the 2nd equation to the 1st equation:

$$\begin{cases} x & - & z & = & -2 \\ & y & - & z & = & 3 \\ & & 0 & = & 0 \end{cases} \iff \begin{cases} x = z - 2 \\ y = z + 3 \end{cases}$$

Here z is a *free variable* (x and y are *leading variables*).

It follows that $\begin{cases} x = t - 2 \\ y = t + 3 \\ z = t \end{cases}$ for some $t \in \mathbb{R}$.

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases}$$

Solution: $(x, y, z) = (t - 2, t + 3, t), \; t \in \mathbb{R}.$

In vector form, $(x, y, z) = (-2, 3, 0) + t(1, 1, 1).$

The set of all solutions is a straight line in $\mathbb{R}^3$ passing through the point $(-2, 3, 0)$ in the direction $(1, 1, 1)$.

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ \phantom{x + {}} y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Add the 1st equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ \phantom{x + {}} y - z = 3 \\ \phantom{x + {}} 5y - 5z = 2 \end{cases}$$

Add -5 times the 2nd equation to the 3rd equation:

$$\begin{cases} x + y - 2z = 1 \\ \phantom{x + {}} y - z = 3 \\ 0 = -13 \end{cases}$$

System of linear equations:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases}$$

Solution: no solution (*inconsistent system*).

Matrices

Definition. A *matrix* is a rectangular array of numbers.

Examples: $\begin{pmatrix} 2 & 7 \\ -1 & 0 \\ 3 & 3 \end{pmatrix}$, $\quad \begin{pmatrix} 2 & 7 & 0.2 \\ 4.6 & 1 & 1 \end{pmatrix}$,

$\begin{pmatrix} 3/5 \\ 5/8 \\ 4 \end{pmatrix}$, $\quad (\sqrt{2}, 0, -\sqrt{3}, 5)$, $\quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

dimensions = (# of rows) $\times$ (# of columns)

$n \times n$: **square matrix**

$n \times 1$: **column vector**

$1 \times n$: **row vector**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix and column vector of the right-hand sides:

$$\begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \qquad \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \qquad \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array} \right)$$

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the augmented matrix.

Elementary operations for systems of linear equations correspond to *elementary row operations* for augmented matrices:

(1) to multiply a row by a nonzero scalar;

(2) to add the ith row multiplied by some $r \in \mathbb{R}$ to the jth row;

(3) to interchange two rows.

Remark. Rows are added and multiplied by scalars as vectors (namely, row vectors).

Elementary row operations

Augmented matrix:

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ \hline a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right) = \left(\begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{array}\right),$$

where $\mathbf{v}_i = (a_{i1} \; a_{i2} \; \dots \; a_{in} \,|\, b_i)$ is a row vector.

Elementary row operations

Operation 1: to multiply the ith row by $r \neq 0$:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 2: to add the *i*th row multiplied by r to the *j*th row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

Elementary row operations

Operation 3: to interchange the ith row with the jth row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$
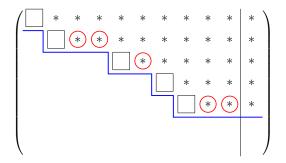
Row echelon form

Definition. **Leading entry** of a matrix is the first nonzero entry in a row.

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:

- leading entries shift to the right as we go from the first row to the last one;
- each leading entry is equal to 1.

$$\left(\begin{array}{cccccccccc|c}
1 & -1 & 3 & 0 & 2 & 1 & 4 & 0 & -3 & 7 & 2 \\
0 & 1 & 1 & -2 & 0 & 0 & 6 & 1 & 3 & 4 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 & 3 & 1 & -4 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 9 & -1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & -3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
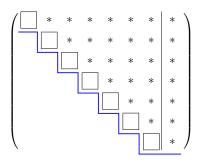0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Row echelon form

General augmented matrix in row echelon form:

- leading entries are boxed (all equal to 1);
- all the entries below the staircase line are zero;
- each step of the staircase has height 1;
- each circle marks a column without a leading entry that corresponds to a free variable.
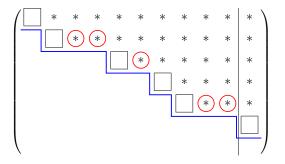
Strict triangular form is a particular case of row echelon form that can occur for systems of n equations in n variables:

$$\left(\begin{array}{ccccccc|c} \square & * & * & * & * & * & * & * \\ & \square & * & * & * & * & * & * \\ & & \square & * & * & * & * & * \\ & & & \square & * & * & * & * \\ & & & & \square & * & * & * \\ & & & & & \square & * & * \\ & & & & & & \square & * \end{array}\right)$$
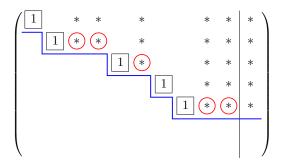
- no zero rows;
- no free variables.

Consistency check

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.

Augmented matrix of an inconsistent system

The goal of the **Gauss-Jordan reduction** is to convert the augmented matrix into **reduced row echelon form**:

- all entries below the staircase line are zero;
- each boxed entry is 1, the other entries in its column are zero;
- each circle corresponds to a free variable.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array} \right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y & = 2 \\ y - z = -1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z = 2 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array}\right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x - z = -2 \\ y - z = 3 \\ 0 = 0 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ \phantom{x + {}} y - z = 3 \\ -x + 4y - 3z = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array}\right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ \phantom{x + {}} y - z = 3 \\ 0 = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

Reduced row echelon form:

$$\begin{cases} x \phantom{{}+ y} - z = 0 \\ \phantom{x + {}} y - z = 0 \\ \phantom{x + y - {}} 0 = 1 \end{cases} \qquad \left(\begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right)$$

How to solve a system of linear equations

- Order the variables.
- Write down the augmented matrix of the system.
- Convert the matrix to **row echelon form**.
- Check for consistency.
- Convert the matrix to **reduced row echelon form**.
- Write down the system corresponding to the reduced row echelon form.
- Determine leading and free variables.
- Rewrite the system so that the leading variables are on the left while everything else is on the right.
- Assign parameters to the free variables and write down the general solution in parametric form.

New example. $\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$

Variables: x_1, x_2, x_3, x_4.

Augmented matrix: $\left(\begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array} \right)$

To get it into row echelon form, we exchange the two rows:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 10 \\ 0 & 1 & 2 & 3 & 6 \end{array} \right)$$

Consistency check is passed. To convert into reduced row echelon form, add -2 times the 2nd row to the 1st row:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array} \right)$$

The leading variables are x_1 and x_2; hence x_3 and x_4 are free variables.

Back to the system:

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

General solution:

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \qquad (t, s \in \mathbb{R})$$

In vector form, $(x_1, x_2, x_3, x_4) =$
$$= (t + 2s - 2, \ -2t - 3s + 6, \ t, \ s)$$
$$= (-2, 6, 0, 0) + t(1, -2, 1, 0) + s(2, -3, 0, 1).$$