

MATH 304
Linear Algebra

Lecture 7:
Inverse matrix (continued).

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1. The $n \times n$ identity matrix is denoted I_n or simply I .

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general,
$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

Theorem. Let A be an arbitrary $m \times n$ matrix. Then $I_m A = A I_n = A$.

Inverse matrix

Definition. Let A be an $n \times n$ matrix. The **inverse** of A is an $n \times n$ matrix, denoted A^{-1} , such that

$$\boxed{AA^{-1} = A^{-1}A = I.}$$

If A^{-1} exists then the matrix A is called **invertible**. Otherwise A is called **singular**.

Basic properties of inverse matrices:

- if $B = A^{-1}$ then $A = B^{-1}$;
- the inverse matrix (if it exists) is unique;
- $(AB)^{-1} = B^{-1}A^{-1}$;
- $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \dots & 0 \\ 0 & d_2^{-1} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_n^{-1} \end{pmatrix}$$

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Theorem A diagonal matrix $D = \text{diag}(d_1, \dots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \text{diag}(d_1^{-1}, \dots, d_n^{-1})$.

Proof: If all $d_i \neq 0$ then, clearly,

$$\text{diag}(d_1, \dots, d_n) \text{diag}(d_1^{-1}, \dots, d_n^{-1}) = \text{diag}(1, \dots, 1) = I,$$

$$\text{diag}(d_1^{-1}, \dots, d_n^{-1}) \text{diag}(d_1, \dots, d_n) = \text{diag}(1, \dots, 1) = I.$$

Now suppose that $d_i = 0$ for some i . Then for any $n \times n$ matrix B the i th row of the matrix DB is a zero row. Hence $DB \neq I$.

Inverting 2×2 matrices

Definition. The **determinant** of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is } \det A = ad - bc.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if

and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad - bc & 0 \\ 0 & ad - bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.

In the case $\det A = 0$, the matrix A is not invertible as

$$\text{otherwise } AB = O \implies A^{-1}(AB) = A^{-1}O = O$$

$$\implies (A^{-1}A)B = O \implies I_2B = O \implies B = O$$

$$\implies A = O, \text{ but the zero matrix is singular.}$$

Problem. Solve a system $\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$

This system is equivalent to a matrix equation $A\mathbf{x} = \mathbf{b}$, where $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

We have $\det A = -1 \neq 0$. Hence A is invertible.

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\implies A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b} \\ &\implies \mathbf{x} = A^{-1}\mathbf{b}. \end{aligned}$$

Conversely, $\mathbf{x} = A^{-1}\mathbf{b} \implies A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}$$

System of n linear equations in n variables:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \iff \mathbf{Ax} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.

General results on inverse matrices

Theorem 1 Given an $n \times n$ matrix A , the following conditions are equivalent:

- (i) A is invertible;
- (ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
- (iii) the matrix equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for any n -dimensional column vector \mathbf{b} ;
- (iv) the row echelon form of A has no zero rows;
- (v) the reduced row echelon form of A is the identity matrix.

Theorem 2 Suppose that a sequence of elementary row operations converts a matrix A into the identity matrix.

Then the same sequence of operations converts the identity matrix into the inverse matrix A^{-1} .

Row echelon form of a square matrix:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case

For any matrix in row echelon form, the number of columns with leading entries equals the number of rows with leading entries. For a square matrix, also the number of columns *without* leading entries (i.e., the number of free variables in a related system of linear equations) equals the number of rows *without* leading entries (i.e., zero rows).

Row echelon form of a square matrix:

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

invertible case

$$\begin{pmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{pmatrix}$$

noninvertible case

Hence the row echelon form of a square matrix A is either strict triangular or else it has a zero row. In the former case, the equation $A\mathbf{x} = \mathbf{b}$ *always* has a unique solution. In the latter case, $A\mathbf{x} = \mathbf{b}$ *never* has a unique solution. Also, in the former case the reduced row echelon form of A is I .

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$

To check whether A is invertible, we convert it to row echelon form.

Interchange the 1st row with the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & -2 & 0 \\ -2 & 3 & 0 \end{pmatrix}$$

Add -3 times the 1st row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Add 2 times the 1st row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & -2 & -3 \\ 0 & 3 & 2 \end{pmatrix}$$

Multiply the 2nd row by -0.5 :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 3 & 2 \end{pmatrix}$$

Add -3 times the 2nd row to the 3rd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & -2.5 \end{pmatrix}$$

Multiply the 3rd row by -0.4 :

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1.5 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{1} & 1.5 \\ 0 & 0 & \boxed{1} \end{pmatrix}$$

We already know that the matrix A is invertible.

Let's proceed towards reduced row echelon form.

Add -1.5 times the 3rd row to the 2nd row:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Add -1 times the 3rd row to the 1st row:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To obtain A^{-1} , we need to apply the following sequence of elementary row operations to the identity matrix:

- interchange the 1st row with the 2nd row,
- add -3 times the 1st row to the 2nd row,
- add 2 times the 1st row to the 3rd row,
- multiply the 2nd row by -0.5 ,
- add -3 times the 2nd row to the 3rd row,
- multiply the 3rd row by -0.4 ,
- add -1.5 times the 3rd row to the 2nd row,
- add -1 times the 3rd row to the 1st row.

A convenient way to compute the inverse matrix A^{-1} is to merge the matrices A and I into one 3×6 matrix $(A | I)$, and apply elementary row operations to this new matrix.

$$A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(A | I) = \left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 3 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Interchange the 1st row with the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 3 & -2 & 0 & 1 & 0 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add -3 times the 1st row to the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ -2 & 3 & 0 & 0 & 0 & 1 \end{array} \right)$$

Add 2 times the 1st row to the 3rd row:

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & -2 & -3 & 1 & -3 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Multiply the 2nd row by -0.5 :

$$\left(\begin{array}{ccc|cc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 3 & 2 & 0 & 2 & 1 \end{array} \right)$$

Add -3 times the 2nd row to the 3rd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & -2.5 & 1.5 & -2.5 & 1 \end{array} \right)$$

Multiply the 3rd row by -0.4 :

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1.5 & -0.5 & 1.5 & 0 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add -1.5 times the 3rd row to the 2nd row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right)$$

Add -1 times the 3rd row to the 1st row:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0.6 & 0 & 0.4 \\ 0 & 1 & 0 & 0.4 & 0 & 0.6 \\ 0 & 0 & 1 & -0.6 & 1 & -0.4 \end{array} \right) = (I | A^{-1})$$

Thus
$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix}.$$

That is,

$$\begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{pmatrix} \frac{3}{5} & 0 & \frac{2}{5} \\ \frac{2}{5} & 0 & \frac{3}{5} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \end{pmatrix} \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$