MATH 304

Linear Algebra

Lecture 15:
Linear independence (continued).
Wronskian.

Linear independence

Definition. Let V be a vector space. Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in V$ are called **linearly dependent** if they satisfy a relation

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where the coefficients $r_1, \dots, r_k \in \mathbb{R}$ are not all equal to zero. Otherwise vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$ are called **linearly independent**. That is, if

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0} \implies r_1 = \cdots = r_k = 0.$$

A set $S \subset V$ is **linearly dependent** if one can find some distinct linearly dependent vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ in S. Otherwise S is **linearly independent**.

Problem Let $\mathbf{v}_1 = (1, 2, 0)$, $\mathbf{v}_2 = (3, 1, 1)$, and $\mathbf{v}_3 = (4, -7, 3)$. Determine whether vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

We have to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + r_3\mathbf{v}_3 = \mathbf{0}$.

This vector equation is equivalent to a system

$$\begin{cases} r_1 + 3r_2 + 4r_3 = 0 \\ 2r_1 + r_2 - 7r_3 = 0 \\ 0r_1 + r_2 + 3r_3 = 0 \end{cases} \qquad \left(\begin{array}{ccr|c} 1 & 3 & 4 & 0 \\ 2 & 1 & -7 & 0 \\ 0 & 1 & 3 & 0 \end{array}\right)$$

The vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent if and only if the coefficient matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is singular. We obtain that $\det A = 0$.

Theorem The following conditions are equivalent:
(i) vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ are linearly dependent;
(ii) one of vectors $\mathbf{v}_1, \dots, \mathbf{v}_k$ is a linear combination of the other $k-1$ vectors.

Proof: **(i)** $\Longrightarrow$ **(ii)** Suppose that

$$r_1\mathbf{v}_1 + r_2\mathbf{v}_2 + \cdots + r_k\mathbf{v}_k = \mathbf{0},$$

where $r_i \neq 0$ for some $1 \leq i \leq k$. Then

$$\mathbf{v}_i = -\tfrac{r_1}{r_i}\mathbf{v}_1 - \cdots - \tfrac{r_{i-1}}{r_i}\mathbf{v}_{i-1} - \tfrac{r_{i+1}}{r_i}\mathbf{v}_{i+1} - \cdots - \tfrac{r_k}{r_i}\mathbf{v}_k.$$

(ii) $\Longrightarrow$ **(i)** Suppose that

$$\mathbf{v}_i = s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k$$

for some scalars s_j. Then

$$s_1\mathbf{v}_1 + \cdots + s_{i-1}\mathbf{v}_{i-1} - \mathbf{v}_i + s_{i+1}\mathbf{v}_{i+1} + \cdots + s_k\mathbf{v}_k = \mathbf{0}.$$

More facts on linear independence

Let S_0 and S be subsets of a vector space V.

- If $S_0 \subset S$ and S is linearly independent, then so is S_0.
- If $S_0 \subset S$ and S_0 is linearly dependent, then so is S.
- If S is linearly independent in V and V is a subspace of W, then S is linearly independent in W.
- The empty set is linearly independent.
- Any set containing $\mathbf{0}$ is linearly dependent.
- Two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if one of them is a scalar multiple the other.
- Two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent if and only if either of them is a scalar multiple the other.
- If S_0 is linearly independent and $\mathbf{v}_0 \in V \setminus S_0$ then $S_0 \cup \{\mathbf{v}_0\}$ is linearly independent if and only if $\mathbf{v}_0 \notin \mathrm{Span}(S_0)$.

Theorem Vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m \in \mathbb{R}^n$ are linearly dependent whenever $m > n$ (i.e., the number of coordinates is less than the number of vectors).

Proof: Let $\mathbf{v}_j = (a_{1j}, a_{2j}, \dots, a_{nj})$ for $j = 1, 2, \dots, m$. Then the vector equality $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_m\mathbf{v}_m = \mathbf{0}$ is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \cdots + a_{1m}t_m = 0, \\ a_{21}t_1 + a_{22}t_2 + \cdots + a_{2m}t_m = 0, \\ \qquad\qquad \cdots\cdots\cdots \\ a_{n1}t_1 + a_{n2}t_2 + \cdots + a_{nm}t_m = 0. \end{cases}$$

Note that vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_m$ are columns of the coefficient matrix (a_{ij}). The number of leading entries in the row echelon form is at most n. If $m > n$ then there are free variables, therefore the zero solution is not unique.

General results on linear independence in $\mathbb{R}^n$

Theorem 1 Given an $n \times m$ matrix A, the following conditions are equivalent:

(i) columns of A are linearly independent (as vectors in $\mathbb{R}^n$);
(ii) $\mathbf{x} = \mathbf{0}$ is the only solution of the matrix equation $A\mathbf{x} = \mathbf{0}$;
(iii) the row echelon form of A has a leading entry in each column.

Theorem 2 Given a square matrix A of dimensions $n \times n$, the following conditions are equivalent:

(i) $\det A \neq 0$;
(ii) columns of A are linearly independent (as vectors in $\mathbb{R}^n$);
(iii) rows of A are linearly independent (as vectors in $\mathbb{R}^n$).

Example. Consider vectors $\mathbf{v}_1 = (1, -1, 1)$, $\mathbf{v}_2 = (1, 0, 0)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (1, 2, 4)$ in $\mathbb{R}^3$.

Two vectors are linearly dependent if and only if they are parallel. Hence $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent if and only if the matrix $A = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is invertible.

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 0 & 1 \end{vmatrix} = - \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = 2 \neq 0.$$

Therefore $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Four vectors in $\mathbb{R}^3$ are always linearly dependent. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent.

Problem. Let $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Determine whether matrices A, A^2, and A^3 are linearly independent.

We have $A = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The task is to check if there exist $r_1, r_2, r_3 \in \mathbb{R}$ not all zero such that $r_1 A + r_2 A^2 + r_3 A^3 = O$.

This matrix equation is equivalent to a system

$$\begin{cases} -r_1 + 0r_2 + r_3 = 0 \\ r_1 - r_2 + 0r_3 = 0 \\ -r_1 + r_2 + 0r_3 = 0 \\ 0r_1 - r_2 + r_3 = 0 \end{cases} \qquad \left(\begin{array}{rrr|r} -1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The row echelon form of the augmented matrix shows there is a free variable. Hence the system has a nonzero solution so that the matrices are linearly dependent (one relation is $A + A^2 + A^3 = O$).

Problem. Show that functions e^x, e^{2x}, and e^{3x} are linearly independent in $C^\infty(\mathbb{R})$.

Suppose that $ae^x + be^{2x} + ce^{3x} = 0$ for all $x \in \mathbb{R}$, where a, b, c are constants. We have to show that $a = b = c = 0$.

Differentiate this identity twice:

$$\begin{aligned} ae^x + be^{2x} + ce^{3x} &= 0, \\ ae^x + 2be^{2x} + 3ce^{3x} &= 0, \\ ae^x + 4be^{2x} + 9ce^{3x} &= 0. \end{aligned}$$

It follows that $A(x)\mathbf{v} = \mathbf{0}$, where

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$A(x) = \begin{pmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

$$\det A(x) = e^x \begin{vmatrix} 1 & e^{2x} & e^{3x} \\ 1 & 2e^{2x} & 3e^{3x} \\ 1 & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^x e^{2x} \begin{vmatrix} 1 & 1 & e^{3x} \\ 1 & 2 & 3e^{3x} \\ 1 & 4 & 9e^{3x} \end{vmatrix}$$

$$= e^x e^{2x} e^{3x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 9 \end{vmatrix}$$

$$= e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 3 & 8 \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 2 \\ 3 & 8 \end{vmatrix} = 2e^{6x} \neq 0.$$

Since the matrix $A(x)$ is invertible, we obtain

$$A(x)\mathbf{v} = \mathbf{0} \implies \mathbf{v} = \mathbf{0} \implies a = b = c = 0$$

Wronskian

Let $f_1, f_2, \ldots, f_n$ be smooth functions on an interval $[a, b]$. The **Wronskian** $W[f_1, f_2, \ldots, f_n]$ is a function on $[a, b]$ defined by

$$W[f_1, f_2, \ldots, f_n](x) = \begin{vmatrix} f_1(x) & f_2(x) & \cdots & f_n(x) \\ f_1'(x) & f_2'(x) & \cdots & f_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & f_2^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{vmatrix}.$$

Theorem If $W[f_1, f_2, \ldots, f_n](x_0) \neq 0$ for some $x_0 \in [a, b]$ then the functions $f_1, f_2, \ldots, f_n$ are linearly independent in $C[a, b]$.