MATH 304

Linear Algebra

Lecture 36:
Orthogonal matrices.
Rigid motions.
Rotations in space.

Orthogonal matrices

Definition. A square matrix A is called **orthogonal** if $AA^T = A^TA = I$, i.e., $A^T = A^{-1}$.

Theorem 1 If A is an $n\times n$ orthogonal matrix, then
(i) columns of A form an orthonormal basis for $\mathbb{R}^n$;
(ii) rows of A also form an orthonormal basis for $\mathbb{R}^n$.

Idea of the proof: Entries of matrix A^TA are dot products of columns of A. Entries of AA^T are dot products of rows of A.

Theorem 2 If A is an $n\times n$ orthogonal matrix, then **(i)** A is diagonalizable in the complexified vector space $\mathbb{C}^n$; **(ii)** all eigenvalues λ of A satisfy $|\lambda| = 1$.

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, $\phi \in \mathbb{R}$.

- $A_\phi A_\psi = A_{\phi+\psi}$
- $A_\phi^{-1} = A_{-\phi} = A_\phi^T$
- A_ϕ is orthogonal
- Eigenvalues: $\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$, $\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.
- Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$, $\mathbf{v}_2 = (1, i)$.
- $\lambda_2 = \overline{\lambda_1}$ and $\mathbf{v}_2 = \overline{\mathbf{v}_1}$.
- Vectors $\frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\frac{1}{\sqrt{2}}\mathbf{v}_2$ form an orthonormal basis for $\mathbb{C}^2$.

Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix.

Theorem The following conditions are equivalent:

(i) $\|L(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$;

(ii) $L(\mathbf{x}) \cdot L(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;

(iii) the transformation L preserves distance between points: $\|L(\mathbf{x}) - L(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$;

(iv) L preserves length of vectors and angle between vectors;

(v) the matrix A is orthogonal;

(vi) the matrix of L relative to any orthonormal basis is orthogonal;

(vii) L maps some orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis;

(viii) L maps any orthonormal basis for $\mathbb{R}^n$ to another orthonormal basis.

Rigid motions

Definition. A transformation $f:\mathbb{R}^n \to \mathbb{R}^n$ is called an **isometry** (or a **rigid motion**) if it preserves distances between points: $\|f(\mathbf{x}) - f(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$.

Examples. • Translation: $f(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0$ is a fixed vector.

• Isometric linear operator: $f(\mathbf{x}) = A\mathbf{x}$, where A is an orthogonal matrix.

• If f_1 and f_2 are two isometries, then the composition $f_2 \circ f_1$ is also an isometry.

Theorem Any isometry $f:\mathbb{R}^n \to \mathbb{R}^n$ can be represented as $f(\mathbf{x}) = A\mathbf{x} + \mathbf{x}_0$, where $\mathbf{x}_0 \in \mathbb{R}^n$ and A is an orthogonal matrix.

Suppose $L:\mathbb{R}^n \to \mathbb{R}^n$ is a linear isometric operator.

Theorem There exists an orthonormal basis for $\mathbb{R}^n$ such that the matrix of L relative to this basis has a diagonal block structure

$$\begin{pmatrix} D_{\pm 1} & O & \dots & O \\ O & R_1 & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \dots & R_k \end{pmatrix},$$

where $D_{\pm 1}$ is a diagonal matrix whose diagonal entries are equal to 1 or -1, and

$$R_j = \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}, \;\; \phi_j \in \mathbb{R}.$$

Classification of linear isometries in $\mathbb{R}^2$:

	$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$	$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
	rotation about the origin	reflection in a line
Determinant:	1	-1
Eigenvalues:	$e^{i\phi}$ and $e^{-i\phi}$	-1 and 1

Classification of linear isometries in $\mathbb{R}^3$*:*

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

$A =$ rotation about a line; $B =$ reflection in a plane; $C =$ rotation about a line combined with reflection in the orthogonal plane.

$\det A = 1$, $\det B = \det C = -1$.

A has eigenvalues 1, $e^{i\phi}$, $e^{-i\phi}$. B has eigenvalues -1, 1, 1. C has eigenvalues -1, $e^{i\phi}$, $e^{-i\phi}$.

Example. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ that acts on the standard basis as follows: $L(\mathbf{e}_1) = \mathbf{e}_2$, $L(\mathbf{e}_2) = \mathbf{e}_3$, $L(\mathbf{e}_3) = -\mathbf{e}_1$.

L maps the standard basis to another orthonormal basis, which implies that L is a rigid motion. The matrix of L relative to the standard basis is $A = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

It is orthogonal, which is another proof that L is isometric.

It follows from the classification that the operator L is either a rotation about an axis, or a reflection in a plane, or the composition of a rotation about an axis with the reflection in the plane orthogonal to the axis.

$\det A = -1 < 0$ so that L reverses orientation. Therefore L is not a rotation. Further, $A^2 \neq I$ so that L^2 is not the identity map. Therefore L is not a reflection.

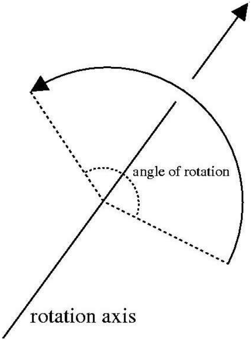
Hence L is a rotation about an axis composed with the reflection in the orthogonal plane. Then there exists an orthonormal basis for $\mathbb{R}^3$ such that the matrix of the operator L relative to that basis is

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

where ϕ is the angle of rotation. Note that the latter matrix is similar to the matrix A. Similar matrices have the same trace (since similar matrices have the same characteristic polynomial and the trace is one of its coefficients). Therefore $\mathrm{trace}(A) = -1 + 2\cos\phi$. On the other hand, $\mathrm{trace}(A) = 0$. Hence $-1 + 2\cos\phi = 0$. Then $\cos\phi = 1/2$ so that $\phi = 60^{\mathrm{o}}$.

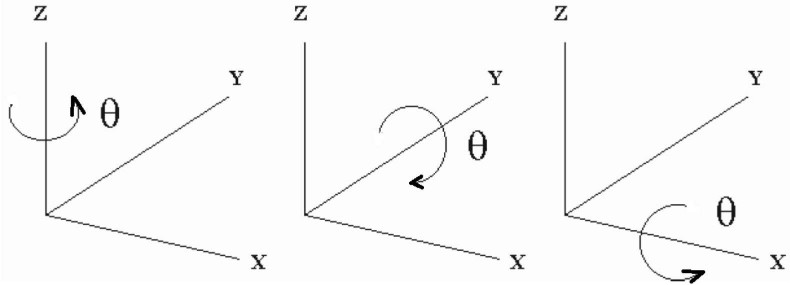
The axis of rotation consists of vectors $\mathbf{v}$ such that $A\mathbf{v} = -\mathbf{v}$. In other words, this is the eigenspace of A associated to the eigenvalue -1. One can find that the eigenspace is spanned by the vector $(1, -1, 1)$.

Rotations in space



If the axis of rotation is oriented, we can say about *clockwise* or *counterclockwise* rotations (with respect to the view from the positive semi-axis).

Counterclockwise rotations about coordinate axes



$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

Problem. Find the matrix of the rotation by 90° about the line spanned by the vector $\mathbf{a} = (1, 2, 2)$. The rotation is assumed to be counterclockwise when looking from the tip of **a**.

$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ is the matrix of (counterclockwise) rotation by 90° about the x-axis.

We need to find an orthonormal basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ such that $\mathbf{v}_1$ points in the same direction as **a**. Also, the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ should obey the same hand rule as the standard basis. Then B will be the matrix of the given rotation relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let U denote the transition matrix from the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (columns of U are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$). Then the desired matrix is $A = UBU^{-1}$.

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is going to be an orthonormal basis, the matrix U will be orthogonal. Then $U^{-1} = U^T$ and $A = UBU^T$.

Remark. The basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the same hand rule as the standard basis if and only if $\det U > 0$.

Hint. Vectors $\mathbf{a} = (1,2,2)$, $\mathbf{b} = (-2,-1,2)$, and $\mathbf{c} = (2,-2,1)$ are orthogonal.

We have $|\mathbf{a}| = |\mathbf{b}| = |\mathbf{c}| = 3$, hence $\mathbf{v}_1 = \frac{1}{3}\mathbf{a}$, $\mathbf{v}_2 = \frac{1}{3}\mathbf{b}$, $\mathbf{v}_3 = \frac{1}{3}\mathbf{c}$ is an orthonormal basis.

Transition matrix: $U = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix}$.

$$\det U = \frac{1}{27}\begin{vmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{vmatrix} = \frac{1}{27} \cdot 27 = 1.$$

(In the case $\det U = -1$, we would change $\mathbf{v}_3$ to $-\mathbf{v}_3$, or change $\mathbf{v}_2$ to $-\mathbf{v}_2$, or interchange $\mathbf{v}_2$ and $\mathbf{v}_3$.)

$$
\begin{aligned}
A &= UBU^T \\
&= \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -2 & -1 & 2 \\ 2 & -2 & 1 \end{pmatrix} \\
&= \frac{1}{9}\begin{pmatrix} 1 & -4 & 8 \\ 8 & 4 & 1 \\ -4 & 7 & 4 \end{pmatrix}.
\end{aligned}
$$