MATH 304

Linear Algebra

Lecture 40:
Review for the final exam (continued).

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.5, 2.1–2.2)

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.
- Linear transformations.
- Matrix of a linear transformation.
- Change of basis for a linear operator.
- Similarity of matrices.

Topics for the final exam: Parts III–IV

Advanced linear algebra (Leon 5.1–5.7, 6.1–6.3)

- Euclidean structure in $\mathbb{R}^n$ (length, angle, dot product).
- Inner products and norms.
- Orthogonal complement, orthogonal projection.
- Least squares problems.
- The Gram-Schmidt orthogonalization process.
- Orthogonal polynomials.

- Eigenvalues, eigenvectors, eigenspaces.
- Characteristic polynomial.
- Bases of eigenvectors, diagonalization.
- Matrix exponentials.
- Complex eigenvalues and eigenvectors.
- Orthogonal matrices.
- Rigid motions, rotations in space.

Problem. Let L denote a linear operator on $\mathbb{R}^3$ that acts on vectors from the standard basis as follows: $L(\mathbf{e}_1) = \mathbf{e}_3$, $L(\mathbf{e}_2) = \mathbf{e}_1$, $L(\mathbf{e}_3) = \mathbf{e}_2$. Describe L in geometric terms.

Alternative solution: The operator L maps one orthonormal basis to an orthonormal basis (namely, the standard basis is mapped to itself). Therefore L is a rigid motion. According to the classification of linear isometries in $\mathbb{R}^3$, L is either a rotation about an axis, or a reflection in a plane, or the composition of two.

Note that $L^3(\mathbf{e}_1) = L(L(L(\mathbf{e}_1))) = L(L(\mathbf{e}_3)) = L(\mathbf{e}_2) = \mathbf{e}_1$. Likewise, $L^3(\mathbf{e}_2) = \mathbf{e}_2$ and $L^3(\mathbf{e}_3) = \mathbf{e}_3$. Since L^3 is linear, it is the identity map. Now it follows that L preserves orientation and so is a rotation. Let ϕ be the angle of rotation, $0 \leq \phi \leq \pi$. Then L^3 is a rotation by 3ϕ. Since L^3 is the identity, we obtain that $3\phi = 2\pi$. The axis of rotation is the line spanned by $(1,1,1)$ since $L(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3) =$
$= L(\mathbf{e}_1) + L(\mathbf{e}_2) + L(\mathbf{e}_3) = \mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2$.

Problem. Find a matrix exponential $\exp(A)$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

One way to find $\exp(A)$ is to diagonalize the matrix A.

Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 2$.

Associated eigenvectors: $\mathbf{v}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Diagonalization: $A = UDU^{-1}$, where

$$D = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}, \qquad U = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Then $e^A = Ue^D U^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^0 & 0 \\ 0 & e^2 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$.

Solution: $\exp(A) = \frac{1}{2} \begin{pmatrix} e^2+1 & e^2-1 \\ e^2-1 & e^2+1 \end{pmatrix}$.

Problem. Consider a system of linear equations in variables x, y, z:

$$\begin{cases} x + 2y - z = 1, \\ 2x + 3y + z = 3, \\ x + 3y + az = 0, \\ x + y + 2z = b. \end{cases}$$

Find values of parameters a and b for which the system has infinitely many solutions, and solve the system for these values.

To determine the number of solutions for the system, we convert its augmented matrix to row echelon form using elementary row operations:

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 2 & 3 & 1 & 3 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 1 & 3 & a & 0 \\ 1 & 1 & 2 & b \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 1 & 1 & 2 & b \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 1 & a+1 & -1 \\ 0 & -1 & 3 & b-1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 1 & a+1 & -1 \\ 0 & -1 & 3 & b-1 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & a+4 & 0 \\ 0 & -1 & 3 & b-1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & a+4 & 0 \\ 0 & 0 & 0 & b-2 \end{array}\right).$$

Now the augmented matrix is in row echelon form (except for the case $a = -4$, $b \neq 2$ when one also needs to exchange the last two rows).

If $b \neq 2$, then there is a leading entry in the rightmost column, which indicates inconsistency.

In the case $b = 2$, the system is consistent. If, additionally, $a \neq -4$ then there is a leading entry in each of the first three columns, which implies uniqueness of the solution.

Thus the system has infinitely many solutions only if $a = -4$ and $b = 2$.

Thus the system has infinitely many solutions only if $a = -4$ and $b = 2$. To find the solutions, we proceed to reduced row echelon form (for these particular values of parameters):

$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The latter matrix is the augmented matrix of the following system of linear equations (which is equivalent to the given one):

$$\begin{cases} x + 5z = 3, \\ y - 3z = -1 \end{cases} \iff \begin{cases} x = -5z + 3, \\ y = 3z - 1. \end{cases}$$

The general solution is $(x, y, z) = (-5t + 3,\ 3t - 1,\ t)$ $= (3, -1, 0) + t(-5, 3, 1),\ \ t \in \mathbb{R}$.

Problem. Let V be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$
\begin{aligned}
f_1'(x) &= (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x), \\
f_2'(x) &= (x \cos x)' = -x \sin x + \cos x \\
&= -f_1(x) + f_4(x), \\
f_3'(x) &= (\sin x)' = \cos x = f_4(x), \\
f_4'(x) &= (\cos x)' = -\sin x = -f_3(x).
\end{aligned}
$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A-\lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2+1)+(\lambda^2+1) = (\lambda^2+1)^2 = (\lambda-i)^2(\lambda+i)^2.$$

The roots are i and $-i$, both of multiplicity 2.

One can show that both eigenspaces of A are one-dimensional. The eigenspace for i is spanned by $(0,0,i,1)$ and the eigenspace for $-i$ is spanned by $(0,0,-i,1)$. It follows that the matrix A is not diagonalizable in $\mathbb{C}^4$.

There is also an indirect way to show that A is not diagonalizable in $\mathbb{C}^4$. Assume the contrary. Then $A = UPU^{-1}$, where U is an invertible matrix with complex entries and

$$P = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

(note that P should have the same characteristic polynomial as A). This would imply that $A^2 = UP^2U^{-1}$. But $P^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

Let us check if $A^2 = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, we have a contradiction. Thus the matrix A is not diagonalizable in $\mathbb{C}^4$.