MATH 304–510 Spring 2017

Sample problems for Test 2

Any problem may be altered or replaced by a different one!

Problem 1. Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{u}) = (\mathbf{u} \cdot \mathbf{v}_1)\mathbf{v}_2,$$

where $\mathbf{v}_1 = (1, 2, -1)$ and $\mathbf{v}_2 = (1, 2, 3)$.

(i) Find a matrix M such that $L(\mathbf{u}) = M\mathbf{u}$ for any column vector $\mathbf{u} \in \mathbb{R}^3$.
(ii) Find all eigenvalues and eigenvectors of L.

Problem 2 Let V be a subspace of $\mathcal{F}(\mathbb{R})$ spanned by functions e^x and e^{-x}. Let L be a linear operator on V such that

$$\begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$$

is the matrix of L relative to the basis e^x, e^{-x}. Find the matrix of L relative to the basis $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$.

Problem 3. Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix A.
(ii) For each eigenvalue of A, find an associated eigenvector.
(iii) Is the matrix A diagonalizable? Explain.
(iv) Find all eigenvalues of the matrix A^2.

Problem 4. Find a linear polynomial which is the best least squares fit to the following data:

x	-2	-1	0	1	2
$f(x)$	-3	-2	1	2	5

Problem 5. Let V be a subspace of $\mathbb{R}^4$ spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

(i) Find an orthonormal basis for V.
(ii) Find an orthonormal basis for the orthogonal complement $V^\perp$.

Problem 6. Let $L : V \to W$ be a linear mapping of a finite-dimensional vector space V to a vector space W. Show that

$$\dim \mathrm{Range}(L) + \dim \ker(L) = \dim V.$$

Problem 7. Prove that every subspace of $\mathbb{R}^n$ is the solution set for some system of linear homogeneous equations in n variables.