Math 311-503 Spring 2007

Sample problems for the final exam: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (20 pts.) The planes $x + 2y + 2z = 1$ and $4x + 7y + 4z = 5$ intersect in a line. Find a parametric representation for the line.

To find the intersection set, we need to solve the system

$$\begin{cases} x + 2y + 2z = 1, \\ 4x + 7y + 4z = 5. \end{cases}$$

Let us convert the system to reduced form using elementary operations:

$$\begin{cases} x + 2y + 2z = 1 \\ 4x + 7y + 4z = 5 \end{cases} \iff \begin{cases} x + 2y + 2z = 1 \\ -y - 4z = 1 \end{cases} \iff \begin{cases} x + 2y + 2z = 1 \\ y + 4z = -1 \end{cases} \iff \begin{cases} x - 6z = 3 \\ y + 4z = -1 \end{cases}$$

It follows that the general solution of the system is $x = 6t + 3$, $y = -4t - 1$, $z = t$, where $t \in \mathbb{R}$. Therefore the two planes intersect in the line $t(6, -4, 1) + (3, -1, 0)$.

Problem 2 (30 pts.) Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 1, 1),\ \mathbf{v}_2 = (1, 2, 2).$$

(i) Find the matrix of the operator L.

Given $\mathbf{v} = (x, y, z) \in \mathbb{R}^3$, we have that $\mathbf{v}\cdot\mathbf{v}_1 = x+y+z$ and $L(\mathbf{v}) = (x+y+z, 2(x+y+z), 2(x+y+z))$. Let A denote the matrix of the linear operator L. The columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$, where $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis for $\mathbb{R}^3$. Therefore

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}.$$

(ii) Find the dimensions of the image and the null-space of L.

The image $\operatorname{Im} L$ of the linear operator L is the subspace of all vectors of the form $L(\mathbf{v})$, where $\mathbf{v} \in \mathbb{R}^3$. It is easy to see that $\operatorname{Im} L$ is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$. Hence $\dim \operatorname{Im} L = 1$.

The null-space $\operatorname{Null} L$ of the operator L is the subspace of all vectors $\mathbf{x} \in \mathbb{R}^3$ such that $L(\mathbf{x}) = \mathbf{0}$. Clearly, $L(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} \cdot \mathbf{v}_1 = 0$. Therefore $\operatorname{Null} L$ is the plane $x + y + z = 0$ orthogonal to $\mathbf{v}_1$ and passing through the origin. Its dimension is 2.

(iii) Find bases for the image and the null-space of L.

Since the image of L is the line spanned by the vector $\mathbf{v}_2 = (1, 2, 2)$, this vector is a basis for the image. The null-space of L is the plane given by the equation $x + y + z = 0$. The general solution of the equation is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. It gives rise to a parametric representation $t(-1, 1, 0) + s(-1, 0, 1)$ of the plane. Thus the null-space of L is spanned by the vectors $(-1, 1, 0)$ and $(-1, 0, 1)$. Since the two vectors are linearly independent, they form a basis for $\operatorname{Null} L$.

Problem 3 (35 pts.) Let $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix A.

The determinant can be evaluated using row or column expansions. For example, let us expand the determinant of A by the first row:

$$\begin{vmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 1 \end{vmatrix} - \begin{vmatrix} 1 & 1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{vmatrix}.$$

Then expand each of the two 3-by-3 determinants by the third row:

$$\det A = \left(\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & -2 \end{vmatrix} \right) - \begin{vmatrix} 1 & 1 \\ 0 & -2 \end{vmatrix} = (0 + (-3)) - (-2) = -1.$$

Another way to evaluate $\det A$ is to reduce the matrix A to the identity matrix using elementary row operations (see below). This requires more work but we are going to do it anyway, to find the inverse of A.

(ii) Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4-by-8 matrix:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Interchange the third row with the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Subtract the second row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & -1 & 1 \end{array}\right).$$

Subtract 2 times the third row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & -1 & 0 & 0 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right).$$

Add the fourth row to the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right).$$

Subtract 2 times the fourth row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -4 & 4 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right).$$

Add 2 times the third row to the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & -4 & 4 & 3 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right).$$

Subtract the second row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & -2 & 2 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & -2 & -1 & 1 \end{array}\right).$$

Finally the left part of our 4-by-8 matrix is transformed into the identity matrix. Therefore the current right side is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & -1 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & -2 & -1 & 0 \\ -2 & 2 & 1 & 0 \\ 1 & -1 & -1 & 1 \\ 2 & -2 & -1 & 1 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of A. We have transformed A into the identity matrix using elementary row operations. These included one row exchange and no row multiplications. It follows that $\det A = -\det I = -1$.

Problem 4 (35 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. One obtains that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 2$$

$$= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1 - \lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3 - \lambda).$$

Hence the matrix B has two eigenvalues: 0 and 3.

(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B?

An eigenvector $\mathbf{x} = (x, y, z)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently, $\mathbf{x} = t(-1, 1, 0) + s(-1, 0, 1)$. Hence the eigenspace of B associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$.

Now consider the case $\lambda = 3$. We obtain that

$$(B - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - y = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = y = z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of B associated with the eigenvalue 3.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix B. They are linearly independent since the matrix whose rows are these vectors is invertible:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B?

It is easy to check that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. To transform the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into an orthogonal one, we only need to orthogonalize the pair $\mathbf{v}_1, \mathbf{v}_2$. Namely, we replace the vector $\mathbf{v}_2$ by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (-1, 0, 1) - \frac{1}{2}(-1, 1, 0) = (-1/2, -1/2, 1).$$

Now $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$ is an orthogonal basis for $\mathbb{R}^3$. Since $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, it is also an eigenvector of B associated with the eigenvalue 0.

Finally, vectors $\mathbf{w}_1 = \frac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\mathbf{w}_2 = \frac{\mathbf{u}}{|\mathbf{u}|}$, and $\mathbf{w}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|}$ form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B. We get that $|\mathbf{v}_1| = \sqrt{2}$, $|\mathbf{u}| = \sqrt{3/2}$, and $|\mathbf{v}_3| = \sqrt{3}$. Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1, 1, 0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1, -1, 2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

Problem 5 (30 pts.) Find a quadratic polynomial that is an orthogonal polynomial relative to the inner product

$$\langle p,q\rangle = \int_0^1 xp(x)q(x)\,dx.$$

First observe that for any integers $m,n\ge 0$,

$$\langle x^m,x^n\rangle = \int_0^1 x^{m+n+1}\,dx = \frac{1}{m+n+2}.$$

To get the first three orthogonal polynomials, we apply the Gram-Schmidt orthogonalization process to the polynomials 1, x, and x^2:

$$\begin{aligned}
p_0(x) &= 1,\\
p_1(x) &= x-\frac{\langle x,p_0\rangle}{\langle p_0,p_0\rangle}p_0(x) = x-\frac{\langle x,1\rangle}{\langle 1,1\rangle} = x-\frac{1/3}{1/2} = x-\frac{2}{3},\\
p_2(x) &= x^2-\frac{\langle x^2,p_0\rangle}{\langle p_0,p_0\rangle}p_0(x)-\frac{\langle x^2,p_1\rangle}{\langle p_1,p_1\rangle}p_1(x) = x^2-\frac{\langle x^2,1\rangle}{\langle 1,1\rangle}-\frac{\langle x^2,p_1\rangle}{\langle p_1,p_1\rangle}\Big(x-\frac{2}{3}\Big).
\end{aligned}$$

Evaluating inner products in the latter formula, we obtain that

$$\begin{aligned}
\frac{\langle x^2,1\rangle}{\langle 1,1\rangle} &= \frac{1/4}{1/2} = \frac{1}{2},\\
\langle x^2,p_1\rangle &= \langle x^2,x\rangle-\frac{2}{3}\langle x^2,1\rangle = \frac{1}{5}-\frac{2}{3}\cdot\frac{1}{4} = \frac{1}{5}-\frac{1}{6} = \frac{1}{30},\\
\langle p_1,p_1\rangle &= \langle x,x\rangle-2\cdot\frac{2}{3}\langle x,1\rangle+\Big(\frac{2}{3}\Big)^2\langle 1,1\rangle = \frac{1}{4}-2\cdot\frac{2}{3}\cdot\frac{1}{3}+\Big(\frac{2}{3}\Big)^2\cdot\frac{1}{2} = \frac{1}{4}-\frac{2}{9} = \frac{1}{36},\\
\frac{\langle x^2,p_1\rangle}{\langle p_1,p_1\rangle} &= \frac{1/30}{1/36} = \frac{6}{5}.
\end{aligned}$$

Thus

$$p_2(x) = x^2-\frac{1}{2}-\frac{6}{5}\Big(x-\frac{2}{3}\Big) = x^2-\frac{6}{5}x-\frac{1}{2}+\frac{4}{5} = x^2-\frac{6}{5}x+\frac{3}{10}$$

is the desired orthogonal polynomial.

Alternative solution: A quadratic polynomial $p(x)=x^2+ax+b$ is an orthogonal polynomial if $\langle p,q\rangle=0$ for any polynomial q such that $\deg q<\deg p$. Actually, it is enough to require that $\langle p,1\rangle=\langle p,x\rangle=0$. Note that

$$\langle p,1\rangle = \int_0^1 (x^3+ax^2+bx)\,dx = \frac{1}{4}+\frac{a}{3}+\frac{b}{2},$$

$$\langle p,x\rangle = \int_0^1 (x^4+ax^3+bx^2)\,dx = \frac{1}{5}+\frac{a}{4}+\frac{b}{3}.$$

Hence $p(x)$ is an orthogonal polynomial if and only if the coefficients a and b satisfy the following system:

$$\begin{cases} a/3+b/2=-1/4,\\ a/4+b/3=-1/5.\end{cases}$$

Solving the system, we obtain

$$\begin{cases} a/3+b/2=-1/4\\ a/4+b/3=-1/5\end{cases} \iff \begin{cases} 2a+3b=-1.5\\ 3a+4b=-2.4\end{cases} \iff \begin{cases} 2a+3b=-1.5\\ a+b=-0.9\end{cases}$$

$$\iff \begin{cases} b = 0.3 \\ a + b = -0.9 \end{cases} \iff \begin{cases} a = -1.2 \\ b = 0.3 \end{cases}$$

Thus $p(x) = x^2 - 1.2x + 0.3$ is an orthogonal polynomial.

Bonus Problem 6 (25 pts.) Let S be the set of all points in $\mathbb{R}^3$ that lie at same distance from the planes $x + 2y + 2z = 1$ and $4x + 7y + 4z = 5$. Show that S is the union of two planes and find these planes.

For any point $\mathbf{x}_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ let $d_1(\mathbf{x}_0)$ denote the distance from $\mathbf{x}_0$ to the plane $x+2y+2z = 1$ and $d_2(\mathbf{x}_0)$ denote the distance from $\mathbf{x}_0$ to the plane $4x + 7y + 4z = 5$. Then

$$d_1(\mathbf{x}_0) = \frac{|x_0 + 2y_0 + 2z_0 - 1|}{\sqrt{1^2 + 2^2 + 2^2}} = \frac{1}{3}\,|x_0 + 2y_0 + 2z_0 - 1|,$$

$$d_2(\mathbf{x}_0) = \frac{|4x_0 + 7y_0 + 4z_0 - 5|}{\sqrt{4^2 + 7^2 + 4^2}} = \frac{1}{9}\,|4x_0 + 7y_0 + 4z_0 - 5|.$$

The point $\mathbf{x}_0$ belongs to the set S if $d_1(\mathbf{x}_0) = d_2(\mathbf{x}_0)$, i.e., if

$$\frac{1}{3}\,|x_0 + 2y_0 + 2z_0 - 1| = \frac{1}{9}\,|4x_0 + 7y_0 + 4z_0 - 5|.$$

This means that

$$3(x_0 + 2y_0 + 2z_0 - 1) = 4x_0 + 7y_0 + 4z_0 - 5$$

or

$$3(x_0 + 2y_0 + 2z_0 - 1) = -(4x_0 + 7y_0 + 4z_0 - 5).$$

Equivalently, $x_0 + y_0 - 2z_0 = 2$ or $7x_0 + 13y_0 + 10z_0 = 8$.

Thus the set S is the union of the planes $x + y - 2z = 2$ and $7x + 13y + 10z = 8$.

Bonus Problem 7 (35 pts.) (i) Find a matrix exponential $\exp(tC)$, where $C = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$ and $t \in \mathbb{R}$.

Observe that $C = 2I + D$, where $D = \begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}$. Then $tC = 2tI + tD$ for all $t \in \mathbb{R}$. Clearly, $(2tI)(tD) = (tD)(2tI) = 2t^2D$. It follows that $\exp(tC) = \exp(2tI)\exp(tD)$. For any square matrix X,

$$\exp(X) = I + X + \frac{1}{2!}X^2 + \cdots + \frac{1}{n!}X^n + \cdots$$

In particular,

$$\exp(2tI) = I + 2tI + \frac{1}{2!}(2tI)^2 + \cdots + \frac{1}{n!}(2tI)^n + \cdots = \left(1 + 2t + \frac{(2t)^2}{2!} + \cdots + \frac{(2t)^n}{n!} + \cdots\right) I = e^{2t}I.$$

Further notice that $D^2 = O$. Then $D^n = O$ for any integer $n \geq 2$. Consequently,

$$\exp(tD) = I + tD = \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix}.$$

Finally,

$$\exp(tC) = \exp(2tI)\exp(tD) = e^{2t}I \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix} = e^{2t} \begin{pmatrix} 1 & 3t \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{2t} & 3te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

(ii) Solve a system of differential equations

$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y, \\ \dfrac{dy}{dt} = 2y \end{cases}$$

subject to the initial conditions $x(0) = y(0) = 1$.

The initial value problem has a unique solution

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{tC}\mathbf{v}_0, \quad \text{where} \quad \mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

By the above

$$e^{tC} = \begin{pmatrix} e^{2t} & 3te^{2t} \\ 0 & e^{2t} \end{pmatrix}.$$

Thus $x(t) = e^{2t}(1 + 3t)$, $y(t) = e^{2t}$.