Math 311-503 Spring 2007

Sample problems for Test 1: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (25 pts.) Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2,0,0)$, $(1,1,0)$, and $(-3,0,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1,1,1)$ in the direction $(2,2,2)$.

(i) Find a parametric representation for the line ℓ.

$t(2,2,2)+(1,1,1)$. Since the line ℓ passes through the origin ($t=-1/2$), an equivalent representation is $s(2,2,2)$.

(ii) Find a parametric representation for the plane Π.

Since the plane Π contains the points $\mathbf{a}=(2,0,0)$, $\mathbf{b}=(1,1,0)$, and $\mathbf{c}=(-3,0,2)$, the vectors $\mathbf{b}-\mathbf{a}=(-1,1,0)$ and $\mathbf{c}-\mathbf{a}=(-5,0,2)$ are parallel to Π. Clearly, $\mathbf{b}-\mathbf{a}$ is not parallel to $\mathbf{c}-\mathbf{a}$. Hence a parametric representation $t_1(\mathbf{b}-\mathbf{a})+t_2(\mathbf{c}-\mathbf{a})+\mathbf{a}=t_1(-1,1,0)+t_2(-5,0,2)+(2,0,0)$.

(iii) Find an equation for the plane Π.

Since the vectors $\mathbf{b}-\mathbf{a}=(-1,1,0)$ and $\mathbf{c}-\mathbf{a}=(-5,0,2)$ are parallel to the plane Π, their cross product $\mathbf{p}=(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ is orthogonal to Π. We have that

$$\mathbf{p}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 0 \\ -5 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ -5 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 1 \\ -5 & 0 \end{vmatrix}\mathbf{k} = 2\mathbf{i}+2\mathbf{j}+5\mathbf{k}=(2,2,5).$$

A point $\mathbf{x}=(x,y,z)$ is in the plane Π if and only if $\mathbf{p}\cdot(\mathbf{x}-\mathbf{a})=0$. This is an equation for the plane. In coordinate form, $2(x-2)+2y+5z=0$ or $2x+2y+5z=4$.

(iv) Find the point where the line ℓ intersects the plane Π.

Let $\mathbf{x}$ be the point of intersection. Then $\mathbf{x}=t_1(-1,1,0)+t_2(-5,0,2)+(2,0,0)$ for some $t_1,t_2\in\mathbb{R}$ and also $\mathbf{x}=s(2,2,2)$ for some $s\in\mathbb{R}$. It follows that

$$\begin{cases} -t_1-5t_2+2=2s, \\ t_1=2s, \\ 2t_2=2s. \end{cases}$$

Solving this system of linear equations, we obtain that $t_1=4/9$, $t_2=s=2/9$. Hence $\mathbf{x}=s(2,2,2)=(4/9,4/9,4/9)$.

(v) Find the angle between the line ℓ and the plane Π.

Let ϕ denote the angle between the vectors $\mathbf{v}=(2,2,2)$ and $\mathbf{p}=(2,2,5)$. Then

$$\cos\phi=\frac{\mathbf{v}\cdot\mathbf{p}}{|\mathbf{v}|\,|\mathbf{p}|}=\frac{2\cdot2+2\cdot2+2\cdot5}{\sqrt{2^2+2^2+2^2}\,\sqrt{2^2+2^2+5^2}}=\frac{18}{\sqrt{12}\,\sqrt{33}}=\frac{3}{\sqrt{11}}.$$

Note that $0<\phi<\pi/2$ as $\cos\phi>0$. Since the vector $\mathbf{v}$ is parallel to the line ℓ while the vector $\mathbf{p}$ is orthogonal to the plane Π, the angle between ℓ and Π is equal to

$$\frac{\pi}{2}-\phi=\frac{\pi}{2}-\arccos\frac{3}{\sqrt{11}}=\arcsin\frac{3}{\sqrt{11}}.$$

(vi) Find the distance from the origin to the plane Π.

The plane Π can be defined by the equation $2x+2y+5z=4$. Hence the distance from a point (x_0,y_0,z_0) to Π is equal to

$$\frac{|2x_0+2y_0+5z_0-4|}{\sqrt{2^2+2^2+5^2}}=\frac{|2x_0+2y_0+5z_0-4|}{\sqrt{33}}.$$

In particular, the distance from the origin to the plane is equal to $\dfrac{4}{\sqrt{33}}$.

Problem 2 (15 pts.) Let $f(x)=a\sin x+b\cos x+c$. Find a, b, and c so that $f(0)=1$, $f'(0)=2$, and $f''(0)=3$.

$f'(x)=a\cos x-b\sin x$, $f''(x)=-a\sin x-b\cos x$. Therefore $f(0)=b+c$, $f'(0)=a$, $f''(0)=-b$. The desired parameters satisfy the system

$$\begin{cases} b+c=1, \\ a=2, \\ -b=3. \end{cases}$$

It follows that $a=2$, $b=-3$, and $c=4$. Thus $f(x)=2\sin x-3\cos x+4$.

Problem 3 (20 pts.) Let $A=\begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}$. Compute the matrices A^2, A^3, and $p(A)$, where $p(x)=2x^2-3x+1$.

$$A^2=\begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}=\begin{pmatrix} 3\cdot3+5\cdot(-2) & 3\cdot5+5\cdot1 \\ -2\cdot3+1\cdot(-2) & -2\cdot5+1\cdot1 \end{pmatrix}=\begin{pmatrix} -1 & 20 \\ -8 & -9 \end{pmatrix},$$

$$A^3=A^2A=\begin{pmatrix} -1 & 20 \\ -8 & -9 \end{pmatrix}\begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}=\begin{pmatrix} -1\cdot3+20\cdot(-2) & -1\cdot5+20\cdot1 \\ -8\cdot3+(-9)\cdot(-2) & -8\cdot5+(-9)\cdot1 \end{pmatrix}=\begin{pmatrix} -43 & 15 \\ -6 & -49 \end{pmatrix},$$

$$p(A)=2A^2-3A+I=2\begin{pmatrix} -1 & 20 \\ -8 & -9 \end{pmatrix}-3\begin{pmatrix} 3 & 5 \\ -2 & 1 \end{pmatrix}+\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\begin{pmatrix} -10 & 25 \\ -10 & -20 \end{pmatrix}.$$

Problem 4 (20 pts.) Let $A=\begin{pmatrix} 5 & -2 & 4 \\ 4 & -3 & 2 \\ -3 & 4 & -1 \end{pmatrix}$. Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 3-by-6 matrix:

$$\left(\begin{array}{rrr|rrr} 5 & -2 & 4 & 1 & 0 & 0 \\ 4 & -3 & 2 & 0 & 1 & 0 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix. To minimize the number of fractional entries, we do not follow the standard elimination procedure.

Add the third row to the first and second rows:

$$\left(\begin{array}{rrr|rrr} 5 & -2 & 4 & 1 & 0 & 0 \\ 4 & -3 & 2 & 0 & 1 & 0 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right)\to\left(\begin{array}{rrr|rrr} 2 & 2 & 3 & 1 & 0 & 1 \\ 4 & -3 & 2 & 0 & 1 & 0 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right)\to\left(\begin{array}{rrr|rrr} 2 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right).$$

Subtract 2 times the second row from the first row:

$$\left(\begin{array}{rrr|rrr} 2 & 2 & 3 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right).$$

Subtract the first row from the second row and then add the first row to the third row:

$$\left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 1 & 0 & 1 & 1 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 0 & -1 & 3 & 2 \\ -3 & 4 & -1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 0 & -1 & 3 & 2 \\ -3 & 4 & 0 & 1 & -2 & 0 \end{array}\right).$$

Add 3 times the second row to the third row:

$$\left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 0 & -1 & 3 & 2 \\ -3 & 4 & 0 & 1 & -2 & 0 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 0 & -1 & 3 & 2 \\ 0 & 7 & 0 & -2 & 7 & 6 \end{array}\right).$$

Divide the third row by 7 and then subtract it from the second row:

$$\left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 0 & -1 & 3 & 2 \\ 0 & 7 & 0 & -2 & 7 & 6 \end{array}\right) \to \left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 1 & 0 & -1 & 3 & 2 \\ 0 & 1 & 0 & -\frac{2}{7} & 1 & \frac{6}{7} \end{array}\right) \to \left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & -\frac{5}{7} & 2 & \frac{8}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & 1 & \frac{6}{7} \end{array}\right).$$

Interchange the first row with the second row and then interchange the second row with the third row:

$$\left(\begin{array}{rrr|rrr} 0 & 0 & 1 & 1 & -2 & -1 \\ 1 & 0 & 0 & -\frac{5}{7} & 2 & \frac{8}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & 1 & \frac{6}{7} \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{5}{7} & 2 & \frac{8}{7} \\ 0 & 0 & 1 & 1 & -2 & -1 \\ 0 & 1 & 0 & -\frac{2}{7} & 1 & \frac{6}{7} \end{array}\right) \to \left(\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{5}{7} & 2 & \frac{8}{7} \\ 0 & 1 & 0 & -\frac{2}{7} & 1 & \frac{6}{7} \\ 0 & 0 & 1 & 1 & -2 & -1 \end{array}\right).$$

Finally the left part of our 3-by-6 matrix is transformed into the identity matrix. Therefore the current right side is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} -\frac{5}{7} & 2 & \frac{8}{7} \\ -\frac{2}{7} & 1 & \frac{6}{7} \\ 1 & -2 & -1 \end{pmatrix}.$$

Problem 5 (20 pts.) Evaluate the following determinants:

$$\begin{vmatrix} 5 & -2 & 4 \\ 4 & -3 & 2 \\ -3 & 4 & -1 \end{vmatrix}, \qquad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix}, \qquad \begin{vmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix}.$$

To evaluate the first determinant, we convert the matrix to upper triangular form by applying elementary row operations. Applying the same row operations as in the solution of Problem 4, we obtain that

$$\begin{vmatrix} 5 & -2 & 4 \\ 4 & -3 & 2 \\ -3 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 4 & -3 & 2 \\ -3 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 2 & 2 & 3 \\ 1 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -3 & 4 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ -3 & 4 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 7 & 0 \end{vmatrix} = 7 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 7 \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = -7 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix} = 7 \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 7.$$

A shorter way is to combine row operations with a row expansion. Once we get the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 4 & -1 \end{pmatrix},$$

it is convenient to expand its determinant by the first row. Thus

$$\begin{vmatrix} 5 & -2 & 4 \\ 4 & -3 & 2 \\ -3 & 4 & -1 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ -3 & 4 & -1 \end{vmatrix} = 1 \cdot \begin{vmatrix} 1 & 1 \\ -3 & 4 \end{vmatrix} = 1 \cdot 4 - 1 \cdot (-3) = 7.$$

The other two matrices are already upper triangular. Therefore

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{vmatrix} = 1 \cdot 4 \cdot 6 = 24, \qquad \begin{vmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 7 \end{vmatrix} = -1 \cdot 5 \cdot 7 = -35.$$

Bonus Problem 6 (25 pts.) Find the volume of the parallelepiped bounded by the following three pairs of parallel planes in $\mathbb{R}^3$:
(1) $x + y = 2$ and $x + y = 4$,
(2) $y + z = 3$ and $y + z = -3$,
(3) $z = 0$ and $z = 5$.

Let Π denote the parallelepiped. Any vertex of Π is the intersection point of some three planes bounding Π. Let $\mathbf{x}_0 = (x_0, y_0, z_0)$ be the intersection point of the planes $x + y = 2$, $y + z = 3$, and $z = 0$. Let $\mathbf{x}_1 = (x_1, y_1, z_1)$ be the intersection point of the planes $x + y = 4$, $y + z = 3$, and $z = 0$. Let $\mathbf{x}_2 = (x_2, y_2, z_2)$ be the intersection point of the planes $x + y = 2$, $y + z = -3$, and $z = 0$. Let $\mathbf{x}_3 = (x_3, y_3, z_3)$ be the intersection point of the planes $x + y = 2$, $y + z = 3$, and $z = 5$.

The points $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ are vertices of the parallelepiped Π. Moreover, the segments $\mathbf{x}_0\mathbf{x}_1$, $\mathbf{x}_0\mathbf{x}_2$, and $\mathbf{x}_0\mathbf{x}_3$ are adjacent edges of the parallelepiped. The coordinates of the vertices $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$ can be found from the following systems of linear equations:

$$\begin{cases} x_0 + y_0 = 2, \\ y_0 + z_0 = 3, \\ z_0 = 0; \end{cases} \qquad \begin{cases} x_1 + y_1 = 4, \\ y_1 + z_1 = 3, \\ z_1 = 0; \end{cases} \qquad \begin{cases} x_2 + y_2 = 2, \\ y_2 + z_2 = -3, \\ z_2 = 0; \end{cases} \qquad \begin{cases} x_3 + y_3 = 2, \\ y_3 + z_3 = 3, \\ z_3 = 5. \end{cases}$$

Solving them we obtain that $\mathbf{x}_0 = (-1, 3, 0)$, $\mathbf{x}_1 = (1, 3, 0)$, $\mathbf{x}_2 = (5, -3, 0)$, and $\mathbf{x}_3 = (4, -2, 5)$. Since vectors $\mathbf{v}_1 = \mathbf{x}_1 - \mathbf{x}_0 = (2, 0, 0)$, $\mathbf{v}_2 = \mathbf{x}_2 - \mathbf{x}_0 = (6, -6, 0)$, and $\mathbf{v}_3 = \mathbf{x}_3 - \mathbf{x}_0 = (5, -5, 5)$ are represented by adjacent edges of the parallelepiped Π, the volume of Π is equal to the absolute value of the following determinant:

$$\begin{vmatrix} 2 & 0 & 0 \\ 6 & -6 & 0 \\ 5 & -5 & 5 \end{vmatrix} = 2 \cdot (-6) \cdot 5 = -60.$$

Thus the volume of Π is 60.