Math 311-503 February 13, 2007

Test 1: Solutions

Problem 1 (30 pts.) Let Π be the plane in $\mathbb{R}^3$ passing through the points $(1,0,0)$, $(0,0,1)$, and $(0,1,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the points $(1,0,1)$ and $(-2,0,-2)$.

(i) Find a parametric representation for the line ℓ.

Since the line ℓ passes through the points $\mathbf{u} = (1,0,1)$ and $\mathbf{w} = (-2,0,-2)$, its direction is determined by the vector $\mathbf{v} = \mathbf{w} - \mathbf{u} = (-3,0,-3)$. The vector $\mathbf{u} = (1,0,1)$ is a scalar multiple of $\mathbf{v}$, hence it also determines the direction of ℓ. This leads to a parametric representation $t\mathbf{u} + \mathbf{u}$. It is easy to see that the line passes through the origin (take $t = -1$). Therefore another representation is $s\mathbf{u} = s(1,0,1)$.

(ii) Find a parametric representation for the plane Π.

Since the plane Π contains the points $\mathbf{a} = (1,0,0)$, $\mathbf{b} = (0,0,1)$, and $\mathbf{c} = (0,1,2)$, the vectors $\mathbf{b} - \mathbf{a} = (-1,0,1)$ and $\mathbf{c} - \mathbf{a} = (-1,1,2)$ are parallel to Π. Clearly, $\mathbf{b} - \mathbf{a}$ is not parallel to $\mathbf{c} - \mathbf{a}$. Hence we get a parametric representation $t_1(\mathbf{b} - \mathbf{a}) + t_2(\mathbf{c} - \mathbf{a}) + \mathbf{a} = t_1(-1,0,1) + t_2(-1,1,2) + (1,0,0)$.

(iii) Find the point where the line ℓ intersects the plane Π.

Let $\mathbf{x}_0$ be the point of intersection. Then $\mathbf{x}_0 = t_1(-1,0,1) + t_2(-1,1,2) + (1,0,0)$ for some $t_1, t_2 \in \mathbb{R}$ and also $\mathbf{x}_0 = s(1,0,1)$ for some $s \in \mathbb{R}$. It follows that

$$\begin{cases} -t_1 - t_2 + 1 = s, \\ t_2 = 0, \\ t_1 + 2t_2 = s. \end{cases}$$

Solving this system of linear equations, we obtain that $t_1 = s = 1/2$, $t_2 = 0$. Hence $\mathbf{x}_0 = s(1,0,1) = (1/2, 0, 1/2)$.

(iv) Determine whether the plane $2x + y + 2z = 9$ is parallel to the plane Π.

The vector $\mathbf{p} = (2,1,2)$ is orthogonal to the plane $2x + y + 2z = 9$. Therefore this plane is parallel to the plane Π if and only if the vectors $\mathbf{b} - \mathbf{a} = (-1,0,1)$ and $\mathbf{c} - \mathbf{a} = (-1,1,2)$ are orthogonal to $\mathbf{p}$. We have that

$$(\mathbf{b} - \mathbf{a}) \cdot \mathbf{p} = (-1,0,1) \cdot (2,1,2) = -1 \cdot 2 + 0 \cdot 1 + 1 \cdot 2 = 0,$$

$$(\mathbf{c} - \mathbf{a}) \cdot \mathbf{p} = (-1,1,2) \cdot (2,1,2) = -1 \cdot 2 + 1 \cdot 1 + 2 \cdot 2 = 3.$$

Thus $\mathbf{c} - \mathbf{a}$ is not orthogonal to $\mathbf{p}$. Consequently, the two planes are not parallel.

Alternative solution: Any plane parallel to the plane $2x + y + 2z = 9$ is given by the equation $2x + y + 2z = c$ or, equivalently, $\mathbf{p} \cdot \mathbf{x} = c$, where $\mathbf{p} = (2,1,2)$, $\mathbf{x} = (x,y,z)$, and c is a constant. Therefore the plane Π is parallel to the plane $2x + y + 2z = 9$ if and only if $\mathbf{p} \cdot \mathbf{a} = \mathbf{p} \cdot \mathbf{b} = \mathbf{p} \cdot \mathbf{c}$. We have that

$$\mathbf{p} \cdot \mathbf{a} = (2,1,2) \cdot (1,0,0) = 2,$$

$$\mathbf{p} \cdot \mathbf{b} = (2,1,2) \cdot (0,0,1) = 2,$$

$$\mathbf{p} \cdot \mathbf{c} = (2,1,2) \cdot (0,1,2) = 5.$$

Since $\mathbf{p} \cdot \mathbf{a} \neq \mathbf{p} \cdot \mathbf{c}$, the two planes are not parallel.

(v) Find the angle between the line ℓ and the plane $2x + y + 2z = 9$.

Let ϕ denote the angle between the vectors $\mathbf{u} = (1, 0, 1)$ and $\mathbf{p} = (2, 1, 2)$. Then

$$\cos\phi = \frac{\mathbf{u}\cdot\mathbf{p}}{|\mathbf{u}|\,|\mathbf{p}|} = \frac{1\cdot 2 + 0\cdot 1 + 1\cdot 2}{\sqrt{1^2+0^2+1^2}\,\sqrt{2^2+1^2+2^2}} = \frac{4}{\sqrt{2}\,\sqrt{9}} = \frac{2\sqrt{2}}{3}.$$

Note that $0 < \phi < \pi/2$ as $\cos\phi > 0$. Besides,

$$\sin\phi = \sqrt{1-\cos^2\phi} = \sqrt{1-\left(\frac{2\sqrt{2}}{3}\right)^2} = \sqrt{1-\frac{8}{9}} = \sqrt{\frac{1}{9}} = \frac{1}{3}.$$

Since the vector $\mathbf{u}$ is parallel to the line ℓ while the vector $\mathbf{p}$ is orthogonal to the plane $2x+y+2z = 9$, the angle between the line and the plane is equal to

$$\frac{\pi}{2} - \phi = \frac{\pi}{2} - \arcsin\frac{1}{3} = \arccos\frac{1}{3}.$$

(vi) Find the distance from the origin to the plane $2x + y + 2z = 9$.

The distance from a point (x_0, y_0, z_0) to the plane $2x + y + 2z = 9$ is equal to

$$\frac{|2x_0 + y_0 + 2z_0 - 9|}{\sqrt{2^2+1^2+2^2}} = \frac{|2x_0 + y_0 + 2z_0 - 9|}{3}.$$

In particular, the distance from the origin to the plane is equal to $\frac{9}{3} = 3$.

Problem 2 (20 pts.) Find a quadratic polynomial $p(x)$ such that $p(1) = 1$, $p(2) = 3$, and $p(3) = 7$.

Let $p(x) = ax^2 + bx + c$. Then $p(1) = a + b + c$, $p(2) = 4a + 2b + c$, and $p(3) = 9a + 3b + c$. The coefficients a, b, and c have to be chosen so that

$$\begin{cases} a+b+c = 1, \\ 4a+2b+c = 3, \\ 9a+3b+c = 7. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a+b+c = 1 \\ 4a+2b+c = 3 \\ 9a+3b+c = 7 \end{cases} \iff \begin{cases} a+b+c = 1 \\ 3a+b = 2 \\ 9a+3b+c = 7 \end{cases} \iff \begin{cases} a+b+c = 1 \\ 3a+b = 2 \\ 8a+2b = 6 \end{cases} \iff \begin{cases} a+b+c = 1 \\ 3a+b = 2 \\ 4a+b = 3 \end{cases}$$

$$\iff \begin{cases} a+b+c = 1 \\ 3a+b = 2 \\ a = 1 \end{cases} \iff \begin{cases} a+b+c = 1 \\ b = -1 \\ a = 1 \end{cases} \iff \begin{cases} c = 1 \\ b = -1 \\ a = 1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - x + 1$.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Compute the matrices A^2, A^3, and $q(A)$, where $q(x) = 2x^2 - 3x + 2$.

$$A^2 = AA = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$$A^3 = A^2A = \begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix},$$

$$q(A) = 2A^2 - 3A + 2I = 2\begin{pmatrix} 1 & 2 & 5 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} - 3\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + 2\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 0 & 5 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & -3 & 4 & -1 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix B.

The determinant can be easily evaluated using column expansions. First we expand the determinant of B by the first column:

$$\begin{vmatrix} 0 & 5 & -1 & 0 \\ 0 & 3 & 0 & 2 \\ 1 & -3 & 4 & -1 \\ 0 & 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 5 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix}.$$

Then we expand this new determinant by the second column:

$$\det B = \begin{vmatrix} 5 & -1 & 0 \\ 3 & 0 & 2 \\ 1 & 0 & 1 \end{vmatrix} = -(-1) \cdot \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ 1 & 1 \end{vmatrix} = 3 \cdot 1 - 2 \cdot 1 = 1.$$

Another way to evaluate $\det B$ is to reduce the matrix B to the identity matrix using elementary row operations (see below). This requires much more work but we are going to do it anyway, to find the inverse of B.

(ii) Find the inverse matrix B^{-1}.

First we merge the matrix B with the identity matrix into one 4-by-8 matrix:

$$\left(\begin{array}{cccc|cccc} 0 & 5 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Interchange the third row with the first row:

$$\left(\begin{array}{cccc|cccc} 0 & 5 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \\ 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 5 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Interchange the fourth row with the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \\ 0 & 5 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 5 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right).$$

Subtract 5 times the second row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 5 & -1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & 0 & 0 & -5 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right).$$

Subtract 3 times the second row from the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & 0 & 0 & -5 \\ 0 & 3 & 0 & 2 & 0 & 1 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & -3 \end{array}\right).$$

Multiply the fourth row by -1:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & -1 & 0 & 1 & 0 & -3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Add 5 times the fourth row to the third row:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -5 & 1 & 0 & 0 & -5 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Subtract the fourth row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 1 & -5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 1 & -5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Add the fourth row to the first row:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & -1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 1 & -5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & 0 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 1 & -5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Multiply the third row by -1:

$$\left(\begin{array}{cccc|cccc} 1 & -3 & 4 & 0 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 1 & -5 & 0 & 10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & -3 & 4 & 0 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & 5 & 0 & -10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Subtract 4 times the third row from the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -3 & 4 & 0 & 0 & -1 & 1 & 3 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & 5 & 0 & -10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & -3 & 0 & 0 & 4 & -21 & 1 & 43 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & 5 & 0 & -10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Add 3 times the second row to the first row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -3 & 0 & 0 & 4 & -21 & 1 & 43 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & 5 & 0 & -10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right) \to \left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 4 & -18 & 1 & 37 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & -1 & 5 & 0 & -10 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 3 \end{array}\right).$$

Finally the left part of our 4-by-8 matrix is transformed into the identity matrix. Therefore the current right side is the inverse matrix of B. Thus

$$B^{-1} = \begin{pmatrix} 4 & -18 & 1 & 37 \\ 0 & 1 & 0 & -2 \\ -1 & 5 & 0 & -10 \\ 0 & -1 & 0 & 3 \end{pmatrix}.$$

As a byproduct, we can evaluate the determinant of B. We have transformed B into the identity matrix using elementary row operations. These included two row exchanges and two row multiplications, both times by -1. It follows that the determinant of B is equal to the determinant of the identity matrix: $\det B = \det I = 1$.

Bonus Problem 5 (25 pts.) Let P be the parallelogram bounded by the following two pairs of parallel lines in $\mathbb{R}^2$: $x+y=1$, $x+y=2$, $2x+3y=0$, and $2x+3y=5$.

(i) Find the vertices of P.

Let $\mathbf{x}_1=(x_1,y_1)$ be the intersection point of the lines $x+y=1$ and $2x+3y=0$. Let $\mathbf{x}_2=(x_2,y_2)$ be the intersection point of the lines $x+y=2$ and $2x+3y=0$. Let $\mathbf{x}_3=(x_3,y_3)$ be the intersection point of the lines $x+y=2$ and $2x+3y=5$. Let $\mathbf{x}_4=(x_4,y_4)$ be the intersection point of the lines $x+y=1$ and $2x+3y=5$.

Clearly, the points $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$, and $\mathbf{x}_4$ are vertices of the parallelogram P. Their coordinates can be found from the following systems of linear equations:

$$\begin{cases} x_1+y_1=1, \\ 2x_1+3y_1=0; \end{cases} \qquad \begin{cases} x_2+y_2=2, \\ 2x_2+3y_2=0; \end{cases} \qquad \begin{cases} x_3+y_3=2, \\ 2x_3+3y_3=5; \end{cases} \qquad \begin{cases} x_4+y_4=1, \\ 2x_4+3y_4=5. \end{cases}$$

Solving them we obtain that $\mathbf{x}_1=(3,-2)$, $\mathbf{x}_2=(6,-4)$, $\mathbf{x}_3=(1,1)$, and $\mathbf{x}_4=(-2,3)$.

(ii) Find the angles of P.

The vertices $\mathbf{x}_1$ and $\mathbf{x}_2$ both lie on the line $2x+3y=0$, hence the segment $\mathbf{x}_1\mathbf{x}_2$ is a side of the parallelogram P. Similarly, the segments $\mathbf{x}_2\mathbf{x}_3$, $\mathbf{x}_3\mathbf{x}_4$, and $\mathbf{x}_1\mathbf{x}_4$ are the other sides of P. Let α be the angle of P at the vertex $\mathbf{x}_1$. Then α is the angle between the vectors $\mathbf{x}_2-\mathbf{x}_1=(3,-2)$ and $\mathbf{x}_4-\mathbf{x}_1=(-5,5)$. It follows that

$$\cos\alpha = \frac{(\mathbf{x}_2-\mathbf{x}_1)\cdot(\mathbf{x}_4-\mathbf{x}_1)}{|\mathbf{x}_2-\mathbf{x}_1|\,|\mathbf{x}_4-\mathbf{x}_1|} = \frac{3\cdot(-5)+(-2)\cdot 5}{\sqrt{3^2+(-2)^2}\,\sqrt{(-5)^2+5^2}} = \frac{-25}{\sqrt{13}\,\sqrt{50}} = -\frac{5}{\sqrt{26}}.$$

Thus two angles of the parallelogram P are equal to $\alpha=\arccos(-5/\sqrt{26})$. The other two angles are equal to $\pi-\alpha=\arccos(5/\sqrt{26})$.

Alternative solution: Each of the lines $x+y=1$, $x+y=2$, $2x+3y=0$, and $2x+3y=5$ contains one side of the parallelogram P. Since the vector $\mathbf{p}_1=(1,1)$ is orthogonal to the lines $x+y=1$ and $x+y=2$ while the vector $\mathbf{p}_2=(2,3)$ is orthogonal to the lines $2x+3y=0$ and $2x+3y=5$, it follows that the angle β between $\mathbf{p}_1$ and $\mathbf{p}_2$ is equal to an angle of the parallelogram. We have that

$$\cos\beta = \frac{\mathbf{p}_1\cdot\mathbf{p}_2}{|\mathbf{p}_1|\,|\mathbf{p}_2|} = \frac{1\cdot 2+1\cdot 3}{\sqrt{1^2+1^2}\,\sqrt{2^2+3^2}} = \frac{5}{\sqrt{2}\,\sqrt{13}} = \frac{5}{\sqrt{26}}.$$

Thus two angles of the parallelogram P are equal to $\beta=\arccos(5/\sqrt{26})$. The other two angles are equal to $\pi-\beta=\arccos(-5/\sqrt{26})$.

(iii) Find the area of P.

Since the vectors $\mathbf{x}_2-\mathbf{x}_1=(3,-2)$ and $\mathbf{x}_4-\mathbf{x}_1=(-5,5)$ are represented by adjacent sides of P, the area of P is the absolute value of the following determinant:

$$\begin{vmatrix} 3 & -2 \\ -5 & 5 \end{vmatrix} = 3\cdot 5-(-2)\cdot(-5) = 15-10 = 5.$$

Thus the area of the parallelogram P is equal to 5.

Alternative solution: Since the vectors $\mathbf{x}_2-\mathbf{x}_1=(3,-2)$ and $\mathbf{x}_4-\mathbf{x}_1=(-5,5)$ are represented by adjacent sides of P and $\alpha=\arccos(-5/\sqrt{26})$ is the angle between these sides, the area of the parallelogram is equal to

$$|\mathbf{x}_2-\mathbf{x}_1|\,|\mathbf{x}_4-\mathbf{x}_1|\,\sin\alpha = \sqrt{13}\,\sqrt{50}\,\sin\alpha = 5\sqrt{26}\,\sin\alpha = 5\sqrt{26}\,\sqrt{1-\cos^2\alpha}$$

$$= 5\sqrt{26}\sqrt{1-\left(-\frac{5}{\sqrt{26}}\right)^2} = 5\sqrt{26}\sqrt{\frac{1}{26}} = 5.$$