

Test 2

Problem 1 (20 pts.) Determine which of the following subsets of \mathbb{R}^3 are subspaces. Briefly explain.

- (i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
- (ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
- (iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
- (iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

Problem 2 (20 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the space of 2-by-2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of the operator L relative to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

- (i) Find a basis for the image of f .
- (ii) Find a basis for the null-space of f .

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

- (i) Find all eigenvalues of the matrix B .
- (ii) For each eigenvalue of B , find an associated eigenvector.
- (iii) Is there a basis for \mathbb{R}^2 consisting of eigenvectors of B ?
- (iv) Find all eigenvalues of the matrix B^2 .

Bonus Problem 5 (25 pts.) Solve the following system of differential equations (find all solutions):

$$\begin{cases} \frac{dx}{dt} = 2x + 3y, \\ \frac{dy}{dt} = x + 4y. \end{cases}$$