Math 311-503 March 29, 2007

Test 2: Solutions

Problem 1 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

It is easy to see that each of the sets S_1, S_2, S_3, and S_4 contains the zero vector $(0, 0, 0)$ and all these sets are closed under scalar multiplication.

The set S_1 is the union of three planes $x = 0$, $y = 0$, and $z = 0$. It is not closed under addition as the following example shows: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1)$.

S_2 is a plane passing through the origin. Obviously, it is closed under addition.

The condition $y^2 + z^2 = 0$ is equivalent to $y = z = 0$. Hence S_3 is a line passing through the origin. It is closed under addition.

Since $y^2 - z^2 = (y - z)(y + z)$, the set S_4 is the union of two planes $y - z = 0$ and $y + z = 0$. The following example shows that S_4 is not closed under addition: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

Thus S_2 and S_3 are subspaces of $\mathbb{R}^3$ while S_1 and S_4 are not.

Problem 2 (20 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the space of 2-by-2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Find the matrix of the operator L relative to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let M_L denote the desired matrix. By definition, M_L is a 4-by-4 matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ relative to the basis E_1, E_2, E_3, E_4. We have that

$$L(E_1) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 0 \end{pmatrix} = 1E_1 + 0E_2 + 3E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 3E_4,$$

$$L(E_3) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 4 & 0 \end{pmatrix} = 2E_1 + 0E_2 + 4E_3 + 0E_4,$$

$$L(E_4) = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 4 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{pmatrix}.$$

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix}.$$

(i) Find a basis for the image of f.

The image of the linear operator f is the subspace of $\mathbb{R}^3$ spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (1, -2, -1)$, $\mathbf{v}_2 = (-1, 1, 0)$, and $\mathbf{v}_3 = (-2, 3, 1)$. The third column is a linear combination of the first two, $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$. Therefore the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the same as the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because they are not parallel. It follows that $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

Alternative solution: The image of f is spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (1, -2, -1)$, $\mathbf{v}_2 = (-1, 1, 0)$, and $\mathbf{v}_3 = (-2, 3, 1)$. To check linear independence of these vectors, we evaluate the determinant of A (using expansion by the third row):

$$\det A = \begin{vmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} -1 & -2 \\ 1 & 3 \end{vmatrix} + 1 \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix} = (-1) \cdot (-1) + 1 \cdot (-1) = 0.$$

Since $\det A = 0$, the columns of the matrix A are linearly dependent. Then the image of f is at most two-dimensional. On the other hand, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because they are not parallel. Hence they span a two-dimensional subspace of $\mathbb{R}^3$. It follows that this subspace coincides with the image of f. Therefore $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

(ii) Find a basis for the null-space of f.

The null-space of f is the set of solutions of the vector equation $A\mathbf{x} = \mathbf{0}$. To solve the equation, we shall convert the matrix A to reduced echelon form. Since the right-hand side of the equation is the zero vector, elementary row operations do not change the solution set.

First we add the first row of the matrix A twice to the second row and once to the third one:

$$\begin{pmatrix} 1 & -1 & -2 \\ -2 & 1 & 3 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix}.$$

Then we subtract the second row from the third row:

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we multiply the second row by -1 and add it to the first row:

$$\begin{pmatrix} 1 & -1 & -2 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that the vector equation $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $x - z = y + z = 0$, where $\mathbf{x} = (x, y, z)$. The general solution of the system is $x = t$, $y = -t$, $z = t$ for an arbitrary $t \in \mathbb{R}$. That is, $\mathbf{x} = (t, -t, t) = t(1, -1, 1)$, where $t \in \mathbb{R}$. Thus the null-space of the linear operator f is the line $t(1, -1, 1)$. The vector $(1, -1, 1)$ is a basis for this line.

Backdoor solution: It is easy to observe that the second column of the matrix A is the sum of the first and the third columns. This implies that the vector $\mathbf{w} = (1, -1, 1)$ is in the null-space of f, that is, $A\mathbf{w} = \mathbf{0}$. Since the image of f has already been shown to be two-dimensional, the null-space of f has to be one-dimensional. It follows that the null-space of f is the line spanned by $\mathbf{w}$. Consequently, the vector $\mathbf{w}$ is a basis for the null-space.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 1 & 4-\lambda \end{vmatrix} = (2-\lambda)(4-\lambda) - 3 \cdot 1 = \lambda^2 - 6\lambda + 5 = (\lambda - 1)(\lambda - 5).$$

Hence the matrix B has two eigenvalues: 1 and 5.

(ii) For each eigenvalue of B, find an associated eigenvector.

An eigenvector $\mathbf{x} = (x, y)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$.

First consider the case $\lambda = 1$. We obtain that

$$(B - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + 3y = 0.$$

The general solution is $x = -3t$, $y = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (-3, 1)$ is an eigenvector of B associated with the eigenvalue 1.

Now consider the case $\lambda = 5$. We obtain that

$$(B - 5I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -3 & 3 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

The general solution is $x = t$, $y = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_2 = (1, 1)$ is an eigenvector of B associated with the eigenvalue 5.

(iii) Is there a basis for $\mathbb{R}^2$ consisting of eigenvectors of B?

The vectors $\mathbf{v}_1 = (-3, 1)$ and $\mathbf{v}_2 = (1, 1)$ are eigenvectors of the matrix B. The two eigenvectors are linearly independent because they are associated with different eigenvalues of B (or simply because they are not parallel). Therefore $\mathbf{v}_1, \mathbf{v}_2$ is a basis for $\mathbb{R}^2$.

Alternatively, the existence of a basis for $\mathbb{R}^2$ consisting of eigenvectors of B already follows from the fact that the matrix B has two distinct eigenvalues.

(iv) Find all eigenvalues of the matrix B^2.

By the above the matrix B has eigenvalues 1 and 5. This means that $B\mathbf{v}_1 = \mathbf{v}_1$ and $B\mathbf{v}_2 = 5\mathbf{v}_2$ for some nonzero vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$. Then

$$B^2\mathbf{v}_1 = B(B\mathbf{v}_1) = B\mathbf{v}_1 = \mathbf{v}_1, \qquad B^2\mathbf{v}_2 = B(B\mathbf{v}_2) = B(5\mathbf{v}_2) = 5B\mathbf{v}_2 = 5(5\mathbf{v}_2) = 25\mathbf{v}_2.$$

Thus $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of the matrix B^2 associated with the eigenvalues 1 and 25, respectively. Since a 2-by-2 matrix has at most 2 eigenvalues, 1 and 25 are the only eigenvalues of B^2.

Bonus Problem 5 (25 pts.) Solve the following system of differential equations (find all solutions):

$$\begin{cases} \dfrac{dx}{dt} = 2x + 3y, \\ \dfrac{dy}{dt} = x + 4y. \end{cases}$$

Introducing a vector function $\mathbf{v}(t) = (x(t), y(t))$, we can rewrite the system in the following way:

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v}, \quad \text{where} \quad B = \begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}.$$

As shown in the solution of Problem 4, there is a basis for $\mathbb{R}^2$ consisting of eigenvectors of the matrix B. Namely, $\mathbf{v}_1 = (-3, 1)$ and $\mathbf{v}_2 = (1, 1)$ are eigenvectors of B associated with the eigenvalues 1 and 5, respectively. Also, $\mathbf{v}_1, \mathbf{v}_2$ is a basis for $\mathbb{R}^2$. It follows that

$$\mathbf{v}(t) = r_1(t)\mathbf{v}_1 + r_2(t)\mathbf{v}_2,$$

where r_1 and r_2 are well-defined scalar functions (coordinates of $\mathbf{v}$ relative to the basis $\mathbf{v}_1, \mathbf{v}_2$). Then

$$\frac{d\mathbf{v}}{dt} = \frac{dr_1}{dt}\mathbf{v}_1 + \frac{dr_2}{dt}\mathbf{v}_2, \qquad B\mathbf{v} = r_1 B\mathbf{v}_1 + r_2 B\mathbf{v}_2 = r_1\mathbf{v}_1 + 5r_2\mathbf{v}_2.$$

As a consequence,

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v} \iff \begin{cases} \dfrac{dr_1}{dt} = r_1, \\ \dfrac{dr_2}{dt} = 5r_2. \end{cases}$$

The general solution of the differential equation $r_1' = r_1$ is $r_1(t) = c_1 e^t$, where c_1 is an arbitrary constant. The general solution of the equation $r_2' = 5r_2$ is $r_2(t) = c_2 e^{5t}$, where c_2 is another arbitrary constant. Therefore the general solution of the equation $\mathbf{v}' = B\mathbf{v}$ is

$$\mathbf{v}(t) = c_1 e^t \mathbf{v}_1 + c_2 e^{5t}\mathbf{v}_2 = c_1 e^t \begin{pmatrix} -3 \\ 1 \end{pmatrix} + c_2 e^{5t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -3c_1 e^t + c_2 e^{5t} \\ c_1 e^t + c_2 e^{5t} \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$. Equivalently,

$$\begin{cases} x(t) = -3c_1 e^t + c_2 e^{5t}, \\ y(t) = c_1 e^t + c_2 e^{5t}. \end{cases}$$