MATH 311-504

Topics in Applied Mathematics

Lecture 2-10:
Matrix of a linear transformation (continued).
Eigenvalues and eigenvectors.

Matrix transformations

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \dots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$\mathbf{y} = A\mathbf{x} \iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\iff \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix} + x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix} + \dots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}$$

Coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The coordinate mapping

$$\textit{vector } \mathbf{v} \;\mapsto\; \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

provides a one-to-one correspondence between V and $\mathbb{R}^n$. Besides, this mapping is linear.

Matrix of a linear transformation

Let V, W be vector spaces and $f : V \to W$ be a linear map.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccc}
V & \xrightarrow{f} & W \\
g_1 \Big\downarrow & & \Big\downarrow g_2 \\
\mathbb{R}^n & \longrightarrow & \mathbb{R}^m
\end{array}
$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. It is represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_m$.

Example. $L:\mathbb{R}^2\to\mathbb{R}^2$, $L\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}$.

The matrix of L with respect to the standard basis $\mathbf{e}_1=(1,0)$, $\mathbf{e}_2=(0,1)$ is $\begin{pmatrix}1&1\\0&1\end{pmatrix}$.

The matrix w.r.t. the basis $\mathbf{v}_1=(3,1)$, $\mathbf{v}_2=(2,1)$ is $\begin{pmatrix}2&1\\-1&0\end{pmatrix}$ since $L(\mathbf{v}_1)=2\mathbf{v}_1-\mathbf{v}_2$, $L(\mathbf{v}_2)=\mathbf{v}_1$.

The matrix w.r.t. the basis $\mathbf{w}_1=(0,1)$, $\mathbf{w}_2=(1,0)$ is $\begin{pmatrix}1&0\\1&1\end{pmatrix}$ since $L(\mathbf{w}_1)=\mathbf{w}_1+\mathbf{w}_2$, $L(\mathbf{w}_2)=\mathbf{w}_2$.

Eigenvalues and eigenvectors

Definition. Let V be a vector space and $L : V \to V$ be a linear operator. A number λ is called an **eigenvalue** of the operator L if $\boxed{L(\mathbf{v}) = \lambda\mathbf{v}}$ for a nonzero vector $\mathbf{v} \in V$. The vector $\mathbf{v}$ is called an **eigenvector** of L associated with the eigenvalue λ.

Remarks. • Alternative notation:
eigenvalue = **characteristic value**,
eigenvector = **characteristic vector**.

• The zero vector is never considered an eigenvector.

• If V is a functional space then eigenvectors are also called **eigenfunctions**.

Example. $L : \mathbb{R}^2 \to \mathbb{R}^2,\ L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = 3\begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence $(1, 0)$ is the eigenvector of L associated with the eigenvalue 2 while $(0, -2)$ is the eigenvector of L associated with the eigenvalue 3.

Remark. Eigenvalues and eigenvectors of a matrix transformation $L : \mathbb{R}^n \to \mathbb{R}^n,\ L(\mathbf{x}) = A\mathbf{x}$ are also called eigenvalues and eigenvectors of the matrix A.

Example. $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence $(1,1)$ is the eigenvector of L associated with the eigenvalue 1 while $(1,-1)$ is the eigenvector of L associated with the eigenvalue -1.

Vectors $\mathbf{v}_1 = (1,1)$ and $\mathbf{v}_2 = (1,-1)$ form a basis for $\mathbb{R}^2$. The matrix of L with respect to this basis is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ since $L(\mathbf{v}_1) = \mathbf{v}_1$, $L(\mathbf{v}_2) = -\mathbf{v}_2$.

Eigenspaces

Let $L : V \to V$ be a linear operator.
For any $\lambda \in \mathbb{R}$, let V_λ denotes the set of all eigenvectors of L associated with the eigenvalue λ.

A vector $\mathbf{v} \in V$ belongs to V_λ if $\mathbf{v} \neq \mathbf{0}$ and $L(\mathbf{v}) = \lambda\mathbf{v}$. Then $(L - \lambda)\mathbf{v} = \mathbf{0}$, where $L - \lambda$ denotes the linear operator $\mathbf{v} \mapsto L(\mathbf{v}) - \lambda\mathbf{v}$.

Thus eigenvectors from V_λ are nonzero vectors from the null-space $\mathrm{Null}(L - \lambda)$.

$\lambda \in \mathbb{R}$ is an eigenvalue of L if $\mathrm{Null}(L - \lambda) \neq \{\mathbf{0}\}$.

If $\mathrm{Null}(L - \lambda) \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of L associated with the eigenvalue λ.

How to find eigenvalues and eigenvectors?

$L : \mathbb{R}^n \to \mathbb{R}^n$, $L(\mathbf{x}) = A\mathbf{x}$, where $A \in \mathcal{M}_{n,n}(\mathbb{R})$.

$(L - \lambda)(\mathbf{x}) = (A - \lambda I)\mathbf{x}$ for all $\lambda \in \mathbb{R}$ and $\mathbf{x} \in \mathbb{R}^n$.

λ is an eigenvalue $\iff$ the matrix $A - \lambda I$ is not invertible $\iff$ $\det(A - \lambda I) = 0$

Definition. $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.

Example. $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix}$$

$$= (a - \lambda)(d - \lambda) - bc$$

$$= \lambda^2 - (a + d)\lambda + (ad - bc).$$

Example. $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$.

$$\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ a_{21} & a_{22} - \lambda & a_{23} \\ a_{31} & a_{32} & a_{33} - \lambda \end{vmatrix}$$

$$= -\lambda^3 + c_1\lambda^2 - c_2\lambda + c_3,$$

where $c_1 = a_{11} + a_{22} + a_{33}$ (the *trace* of A),

$$c_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} + \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},$$

$c_3 = \det A$.

Example. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Characteristic equation: $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$.

$(2-\lambda)^2 - 1 = 0 \implies \lambda_1 = 1, \ \lambda_2 = 3.$

$$(A - I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x + y = 0.$$

The general solution is $(-t, t) = t(-1, 1), \ t \in \mathbb{R}$. Thus $\mathbf{v}_1 = (-1, 1)$ is an eigenvector associated with the eigenvalue 1. The corresponding eigenspace is the line spanned by $\mathbf{v}_1$.

$$(A - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff x - y = 0.$$

The general solution is $(t, t) = t(1, 1), \ \ t \in \mathbb{R}$.

Thus $\mathbf{v}_2 = (1, 1)$ is an eigenvector associated with the eigenvalue 3. The corresponding eigenspace is the line spanned by $\mathbf{v}_2$.

Summary. $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- The matrix A has two eigenvalues: 1 and 3.

- The eigenspace of A associated with the eigenvalue 1 is the line $t(-1, 1)$.

- The eigenspace of A associated with the eigenvalue 3 is the line $t(1, 1)$.

- Eigenvectors $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 1)$ of the matrix A form an orthogonal basis for $\mathbb{R}^2$.

- Geometrically, the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a stretch by a factor of 3 away from the line $x + y = 0$ in the orthogonal direction.