MATH 311-504

Topics in Applied Mathematics

Lecture 2-9:
Basis and dimension (continued).
Matrix of a linear transformation.

Basis and dimension

Definition. Let V be a vector space. A linearly independent spanning set for V is called a **basis**.

Theorem Any vector space V has a basis. If V has a finite basis, then all bases for V are finite and have the same number of elements.

Definition. The **dimension** of a vector space V, denoted $\dim V$, is the number of elements in any of its bases.

Examples. • $\dim \mathbb{R}^n = n$

• $\mathcal{M}_{m,n}(\mathbb{R})$: the space of $m \times n$ matrices
$\dim \mathcal{M}_{m,n}(\mathbb{R}) = mn$

• $\mathcal{P}_n$: polynomials of degree at most n
$\dim \mathcal{P}_n = n + 1$

• $\mathcal{P}$: the space of all polynomials
$\dim \mathcal{P} = \infty$

• $\{\mathbf{0}\}$: the trivial vector space
$\dim \{\mathbf{0}\} = 0$

Bases for $\mathbb{R}^n$

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in $\mathbb{R}^n$.

Theorem 1 If $m < n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ do not span $\mathbb{R}^n$.

Theorem 2 If $m > n$ then the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly dependent.

Theorem 3 If $m = n$ then the following conditions are equivalent:
(i) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$;
(ii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a spanning set for $\mathbb{R}^n$;
(iii) $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a linearly independent set.

Theorem Let S be a subset of a vector space V. Then the following conditions are equivalent:

(i) S is a linearly independent spanning set for V, i.e., a basis;

(ii) S is a minimal spanning set for V;

(iii) S is a maximal linearly independent subset of V.

"Minimal spanning set" means "remove any element from this set, and it is no longer a spanning set".

"Maximal linearly independent subset" means "add any element of V to this set, and it will become linearly dependent".

How to find a basis?

Theorem Let V be a vector space. Then
(i) any spanning set for V can be reduced to a minimal spanning set;
(ii) any linearly independent subset of V can be extended to a maximal linearly independent set.

That is, any spanning set contains a basis, while any linearly independent set is contained in a basis.

Approach 1. Get a spanning set for the vector space, then reduce this set to a basis.

Approach 2. Build a maximal linearly independent set adding one vector at a time.

Example. $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(\mathbf{x}) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \mathbf{x}$.

Find the dimension of the image of f.

The image of f is spanned by columns of the matrix: $\mathbf{v}_1 = (1, 2)$, $\mathbf{v}_2 = (1, 1)$, and $\mathbf{v}_3 = (-1, 0)$. Observe that $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$. It follows that $\mathrm{Im}\, f$ is spanned by vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ alone. Clearly, $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. Hence $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathrm{Im}\, f$ and $\dim \mathrm{Im}\, f = 2$.

Alternatively, since $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent, they constitute a basis for $\mathbb{R}^2$.
It follows that $\mathrm{Im}\, f = \mathbb{R}^2$ and $\dim \mathrm{Im}\, f = 2$.

Example. $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f(\mathbf{x}) = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \mathbf{x}$.

Find the dimension of the null-space of f.

The null-space of f is the solution set of the system
$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \mathbf{x} = \mathbf{0}.$$

To solve the system, we convert the matrix to reduced form:
$$\begin{pmatrix} 1 & 1 & -1 \\ 2 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} \color{red}{1} & 0 & 1 \\ 0 & \color{red}{1} & -2 \end{pmatrix}$$

Hence $(x, y, z) \in \mathrm{Null}\, f$ if $x + z = y - 2z = 0$.
General solution: $(x, y, z) = (-t, 2t, t)$, $t \in \mathbb{R}$.
Thus $\mathrm{Null}\, f$ is the line $t(-1, 2, 1)$ and $\dim \mathrm{Null}\, f = 1$.

Example. $L : \mathcal{P}_4 \to \mathcal{P}_4$, $(Lp)(x) = p(x) + p(-x)$. Find the dimensions of $\mathrm{Im}\, L$ and $\mathrm{Null}\, L$.

$p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$
$\implies (Lp)(x) = 2a_0 + 2a_2 x^2 + 2a_4 x^4$.

Since $\{1, x, x^2, x^3, x^4\}$ is a basis for $\mathcal{P}_4$, the image of L is spanned by polynomials $L1, Lx, Lx^2, Lx^3, Lx^4$.
$L1 = 2$, $Lx^2 = 2x^2$, $Lx^4 = 2x^4$, $Lx = Lx^3 = 0$.
Hence $\mathrm{Im}\, L$ is spanned by $1, x^2, x^4$. Clearly, $1, x^2, x^4$ are linearly independent so that they form a basis for $\mathrm{Im}\, L$ and $\dim \mathrm{Im}\, L = 3$.

The null-space of L consists of polynomials $a_1 x + a_3 x^3$. That is, it is spanned by x and x^3. Thus $\{x, x^3\}$ is a basis for $\mathrm{Null}\, L$ and $\dim \mathrm{Null}\, L = 2$.

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \ \mapsto \ \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

provides a one-to-one correspondence between V and $\mathbb{R}^n$. Besides, this mapping is linear.

Matrix of a linear mapping

Let V, W be vector spaces and $f : V \to W$ be a linear map.
Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.
Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ g_1 \big\downarrow & & \big\downarrow g_2 \\ \mathbb{R}^n & \longrightarrow & \mathbb{R}^m \end{array}$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$.
It is represented as $\mathbf{v} \mapsto A\mathbf{v}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \dots, \mathbf{v}_n$ and $\mathbf{w}_1, \dots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \dots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \dots, \mathbf{w}_m$.

Examples. • $D : \mathcal{P}_2 \to \mathcal{P}_1$, $(Dp)(x) = p'(x)$.

Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and $1, x$. Columns of A_D are coordinates of polynomials $D1$, Dx, Dx^2 w.r.t. the basis $1, x$.

$$D1 = 0,\ Dx = 1,\ Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_2 \to \mathcal{P}_2$, $(Lp)(x) = p(x+1)$.

Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$.
$L1 = 1$, $Lx = 1 + x$, $Lx^2 = (x+1)^2 = 1 + 2x + x^2$.

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem. Consider a linear operator $L : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,

$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis $\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let N be the desired matrix. Columns of N are coordinates of the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$.

$$L(\mathbf{v}_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad L(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Clearly, $L(\mathbf{v}_2) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2$.

$$L(\mathbf{v}_1) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \iff \begin{cases} 3\alpha + 2\beta = 4 \\ \alpha + \beta = 1 \end{cases} \iff \begin{cases} \alpha = 2 \\ \beta = -1 \end{cases}$$

Thus $N = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.