Math 311-504

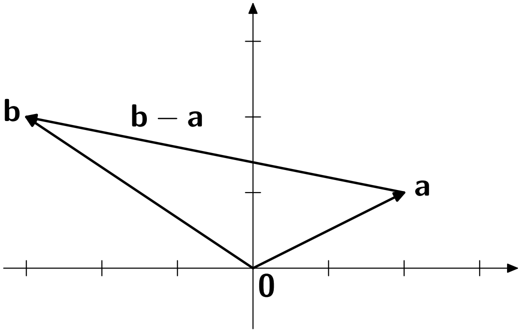
Topics in Applied Mathematics

Lecture 2:
Orthogonal projection.
Lines and planes.

Vectors

n-dimensional vector is an element of $\mathbb{R}^n$, that is, an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ of real numbers.

Elements of $\mathbb{R}^n$ are regarded either as vectors or as points. If $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ are points, then the directed segment $\overrightarrow{\mathbf{ab}}$ represents the vector $\mathbf{b} - \mathbf{a}$.
In particular, each point $\mathbf{a} \in \mathbb{R}^n$ has the same coordinates as its *position vector* $\overrightarrow{\mathbf{0a}}$.



$$\mathbf{a} = (2, 1), \quad \mathbf{b} = (-3, 2), \quad \mathbf{b} - \mathbf{a} = (-5, 1)$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be n-dimensional vectors.

Addition: $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.
Scalar multiplication: $r\mathbf{x} = (rx_1, rx_2, \ldots, rx_n)$.

Length: $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.
Dot product: $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + \cdot + x_ny_n$.
Angle: $\angle(\mathbf{x}, \mathbf{y}) = \arccos \dfrac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|\,|\mathbf{y}|}$.

Problem. Find the angle θ between vectors $\mathbf{x} = (2, -1)$ and $\mathbf{y} = (3, 1)$.

$\mathbf{x} \cdot \mathbf{y} = 5$, $|\mathbf{x}| = \sqrt{5}$, $|\mathbf{y}| = \sqrt{10}$.

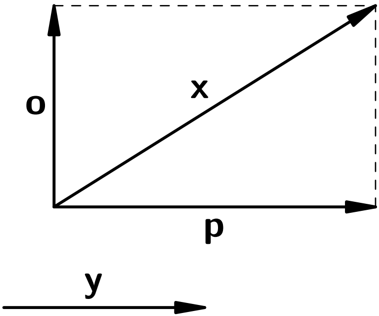
$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|\,|\mathbf{y}|} = \frac{5}{\sqrt{5}\,\sqrt{10}} = \frac{1}{\sqrt{2}} \implies \theta = 45^{\mathrm{o}}$$

Problem. Find the angle ϕ between vectors $\mathbf{v} = (-2, 1, 3)$ and $\mathbf{w} = (4, 5, 1)$.

$$\mathbf{v} \cdot \mathbf{w} = 0 \implies \mathbf{v} \perp \mathbf{w} \implies \phi = 90^{\mathrm{o}}$$

Orthogonal projection

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, with $\mathbf{y} \neq \mathbf{0}$.
Then there exists a unique decomposition $\mathbf{x} = \mathbf{p} + \mathbf{o}$ such that $\mathbf{p}$ is parallel to $\mathbf{y}$ and $\mathbf{o}$ is orthogonal to $\mathbf{y}$.



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Namely, $\mathbf{p} = \alpha\mathbf{u}$, where $\mathbf{u}$ is the unit vector of the same direction as $\mathbf{y}$, and $\alpha = \mathbf{x} \cdot \mathbf{u}$.

Indeed, $\mathbf{p} \cdot \mathbf{u} = (\alpha\mathbf{u}) \cdot \mathbf{u} = \alpha(\mathbf{u} \cdot \mathbf{u}) = \alpha|\mathbf{u}|^2 = \alpha = \mathbf{x} \cdot \mathbf{u}$.
Hence $\mathbf{o} \cdot \mathbf{u} = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{u} = \mathbf{x} \cdot \mathbf{u} - \mathbf{p} \cdot \mathbf{u} = 0 \implies \mathbf{o} \perp \mathbf{u}$
$\implies \mathbf{o} \perp \mathbf{y}$.

$\mathbf{p}$ is called the **vector projection** of $\mathbf{x}$ onto $\mathbf{y}$,
$\alpha = \pm|\mathbf{p}|$ is called the **scalar projection** of $\mathbf{x}$ onto $\mathbf{y}$.

$$\mathbf{u} = \frac{\mathbf{y}}{|\mathbf{y}|}, \qquad \alpha = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{y}|}, \qquad \mathbf{p} = \frac{\mathbf{x} \cdot \mathbf{y}}{\mathbf{y} \cdot \mathbf{y}}\,\mathbf{y}.$$

Lines

A line is specified by one point and a direction.
The direction is specified by a nonzero vector.

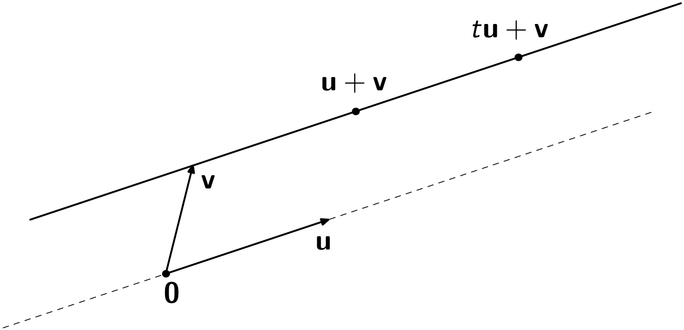
Definition. A *line* is a set of all points $t\mathbf{u} + \mathbf{v}$, where $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v}$ are fixed vectors while t ranges over all real numbers.

Here $\mathbf{v}$ is a point on the line, $\mathbf{u}$ is the direction.
$t\mathbf{u} + \mathbf{v}$ is a *parametric representation* of the line.

Example. $t(1,3,1) + (-2,0,3)$ is a line in $\mathbb{R}^3$.

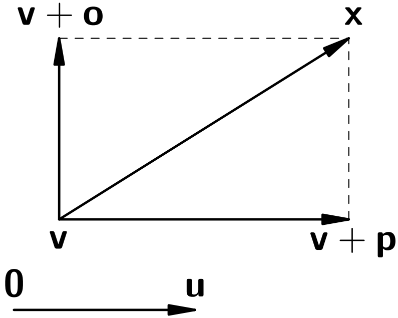
If (x, y, z) is a point on the line, then

$$\begin{cases} x = t - 2, \\ y = 3t, \\ z = t + 3 \end{cases} \qquad \text{for some } t \in \mathbb{R}.$$



Line $t\mathbf{u}+\mathbf{v}$

Problem. Let ℓ denote a line $t\mathbf{u}+\mathbf{v}$.
(i) Find the distance from a point $\mathbf{x}$ to ℓ.
(ii) Find the point on the line ℓ that is closest to $\mathbf{x}$.



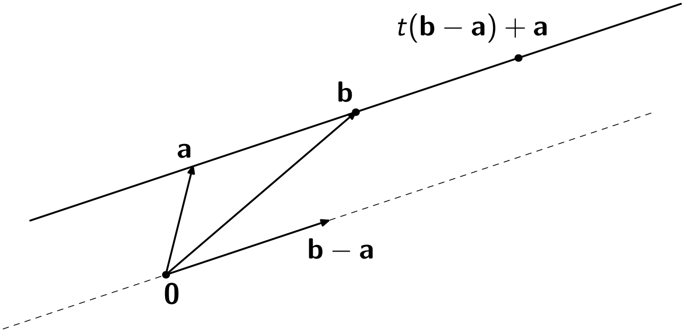
$\mathbf{p}$ = orthogonal projection of $\mathbf{x}-\mathbf{v}$ onto $\mathbf{u}$.
The distance equals $|\mathbf{o}|$. The closest point is $\mathbf{v}+\mathbf{p}$.

Alternatively, a line is specified by two distinct points $\mathbf{a}$ and $\mathbf{b}$. Then the vector $\mathbf{b} - \mathbf{a}$ is parallel to the line, hence $t(\mathbf{b} - \mathbf{a}) + \mathbf{a}$ is a parametric representation.

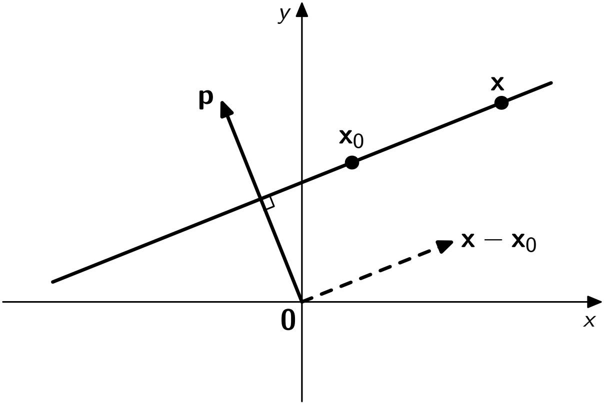
Let $\mathbf{x} = t(\mathbf{b} - \mathbf{a}) + \mathbf{a}$.
Then $\mathbf{x}$ lies between $\mathbf{a}$ and $\mathbf{b}$ if $0 < t < 1$.
If $t > 1$ then $\mathbf{b}$ lies between $\mathbf{a}$ and $\mathbf{x}$.
If $t < 0$ then $\mathbf{a}$ lies between $\mathbf{x}$ and $\mathbf{b}$.

Definition. The *segment* joining points $\mathbf{a}$ and $\mathbf{b}$ is the set of all points $t(\mathbf{b} - \mathbf{a}) + \mathbf{a}$, where $0 \leq t \leq 1$.

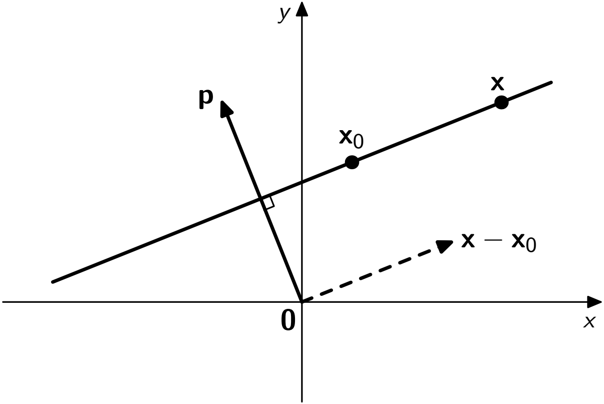
Note that $t(\mathbf{b} - \mathbf{a}) + \mathbf{a} = (1 - t)\mathbf{a} + t\mathbf{b}$.



Line through $\mathbf{a}$ and $\mathbf{b}$



In $\mathbb{R}^2$, a line can also be specified by one point and an orthogonal direction.



Line through $\mathbf{x}_0$ orthogonal to $\mathbf{p}$

$\mathbf{x}$ is on line $\iff$ $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$

Proposition Let $\ell \subset \mathbb{R}^2$ be the line passing through a point $\mathbf{x}_0$ and orthogonal to a vector $\mathbf{p} \neq \mathbf{0}$. Then a point $\mathbf{x} \in \mathbb{R}^2$ is on ℓ if and only if $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.

Suppose $\mathbf{p} = (a, b)$, $\mathbf{x} = (x, y)$, and $\mathbf{x}_0 = (x_0, y_0)$. Then the equation of the line ℓ becomes

$$a(x - x_0) + b(y - y_0) = 0$$

or

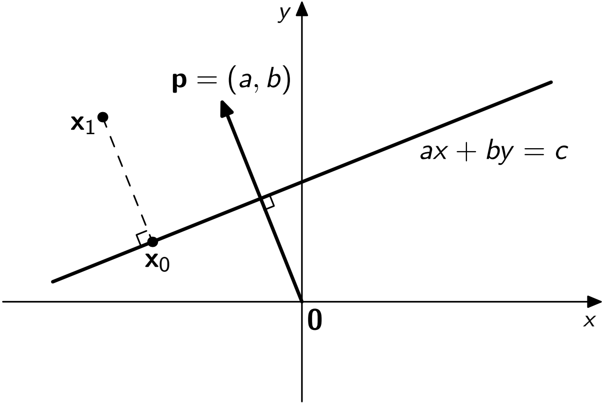
$$ax + by = c, \quad \text{where} \quad c = ax_0 + by_0.$$

Distance to a line in a plane

Proposition Suppose ℓ is a line in $\mathbb{R}^2$ given by the equation $ax + by = c$. Then
(i) the distance from a point (x_1, y_1) to the line ℓ equals

$$\frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}};$$

(ii) two points (x_1, y_1) and (x_2, y_2) are on the same side of ℓ if and only if the numbers $ax_1 + by_1 - c$ and $ax_2 + by_2 - c$ have the same sign.



Distance from $\mathbf{x}_1$ to ℓ is equal to $|\mathbf{x}_1 - \mathbf{x}_0|$

Vector $\mathbf{x}_1 - \mathbf{x}_0$ is parallel to $\mathbf{p}$

Proof of (i)

The vector $\mathbf{p} = (a, b)$ is orthogonal to the line ℓ. The equation $ax + by = c$ can be rewritten as $\mathbf{p} \cdot \mathbf{x} = c$, where $\mathbf{x} = (x, y)$.

Given a point $\mathbf{x}_1 = (x_1, y_1)$, let $\mathbf{x}_0$ be its orthogonal projection on ℓ. Then the distance $dist(\mathbf{x}_1, \ell)$ is equal to $|\mathbf{x}_1 - \mathbf{x}_0|$.

Since vectors $\mathbf{x}_1 - \mathbf{x}_0$ and $\mathbf{p}$ are parallel, $\mathbf{p} \cdot (\mathbf{x}_1 - \mathbf{x}_0) = \pm|\mathbf{p}|\,|\mathbf{x}_1 - \mathbf{x}_0|$.

$$dist = \frac{|\mathbf{p} \cdot (\mathbf{x}_1 - \mathbf{x}_0)|}{|\mathbf{p}|} = \frac{|\mathbf{p} \cdot \mathbf{x}_1 - \mathbf{p} \cdot \mathbf{x}_0|}{|\mathbf{p}|} = \frac{|ax_1 + by_1 - c|}{\sqrt{a^2 + b^2}}$$

Planes

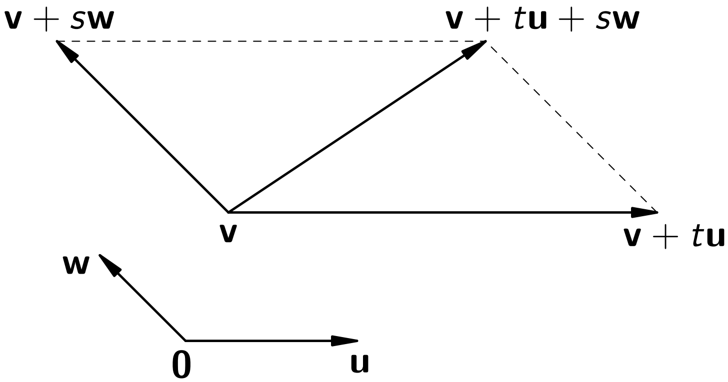
A plane is specified by two intersecting lines.

Definition. A *plane* is a set of all points $t\mathbf{u} + s\mathbf{w} + \mathbf{v}$, where $\mathbf{u}$, $\mathbf{w}$, and $\mathbf{v}$ are fixed vectors such that $\mathbf{u}$ and $\mathbf{w}$ are not parallel, while t and s range over all real numbers.

The plane $t\mathbf{u} + s\mathbf{w} + \mathbf{v}$ contains lines $t\mathbf{u} + \mathbf{v}$ and $s\mathbf{w} + \mathbf{v}$ that intersect at the point $\mathbf{v}$.

$t\mathbf{u} + s\mathbf{w} + \mathbf{v}$ is a *parametric representation*.

Planes



$\mathbf{v} + s\mathbf{w}$
$\mathbf{v} + t\mathbf{u} + s\mathbf{w}$
$\mathbf{v}$
$\mathbf{v} + t\mathbf{u}$
$\mathbf{w}$
$\mathbf{0}$
$\mathbf{u}$

Alternatively, a plane is specified by a line $t\mathbf{u} + \mathbf{v}$ and a point $\mathbf{a}$ outside it. Then a parametric representation is $t\mathbf{u} + s(\mathbf{a} - \mathbf{v}) + \mathbf{v}$.

Alternatively, a plane is specified by three points $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ that are not on the same line. Then a parametric representation is

$$t(\mathbf{b} - \mathbf{a}) + s(\mathbf{c} - \mathbf{a}) + \mathbf{a}$$
$$= (1 - t - s)\mathbf{a} + t\mathbf{b} + s\mathbf{c}.$$

In $\mathbb{R}^3$, a plane can also be specified by one point $\mathbf{x}_0$ and an orthogonal direction $\mathbf{p} \neq \mathbf{0}$. Then the plane is given by the equation $\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}_0) = 0$.

Let $\mathbf{p} = (a, b, c)$, $\mathbf{x} = (x, y, z)$, and $\mathbf{x}_0 = (x_0, y_0, z_0)$. Then the equation of the plane becomes

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

or

$$ax + by + cz = d, \quad \text{where} \quad d = ax_0 + by_0 + xz_0.$$

Distance to a plane in space

Proposition Suppose Π is a plane in $\mathbb{R}^3$ given by the equation $ax + by + cz = d$. Then

(i) the distance from a point (x_1, y_1, z_1) to the plane Π equals

$$\frac{|ax_1 + by_1 + cz_1 - d|}{\sqrt{a^2 + b^2 + c^2}};$$

(ii) two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are on the same side of Π if and only if the numbers $ax_1 + by_1 + cz_1 - d$ and $ax_2 + by_2 + cz_2 - d$ have the same sign.