MATH 311-504

Topics in Applied Mathematics

Lecture 3-12:
Fourier series (continued).

Fourier series

Definition. **Fourier series** is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

To each integrable function $F : [-\pi, \pi] \to \mathbb{R}$ we associate a Fourier series such that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx\, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx\, dx.$$

Convergence in the mean

Theorem Fourier series of a continuous function on $[-\pi, \pi]$ converges to this function with respect to the distance

$$\operatorname{dist}(f, g) = \|f - g\| = \left(\int_{-\pi}^{\pi} |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

However such convergence **in the mean** does not necessarily imply pointwise convergence.

Questions

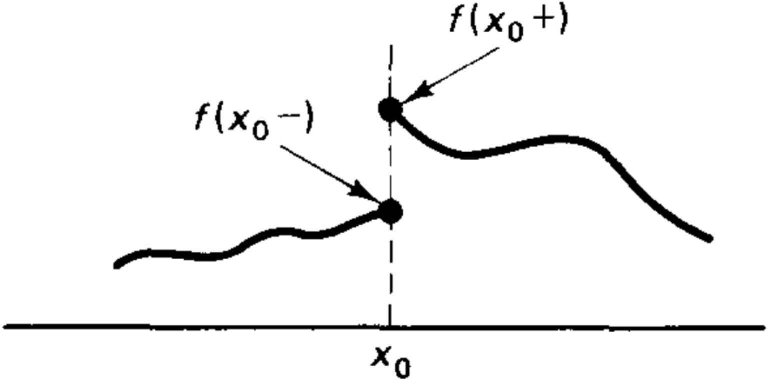
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

- When does a Fourier series converge everywhere? When does it converge uniformly?
- If a Fourier series does not converge everywhere, then what is the set of points where it converges?
- If a Fourier series is associated to a function, then how do convergence properties depend on the function?
- If a Fourier series is associated to a function, then how does the sum of the series relate to the function?

Answers

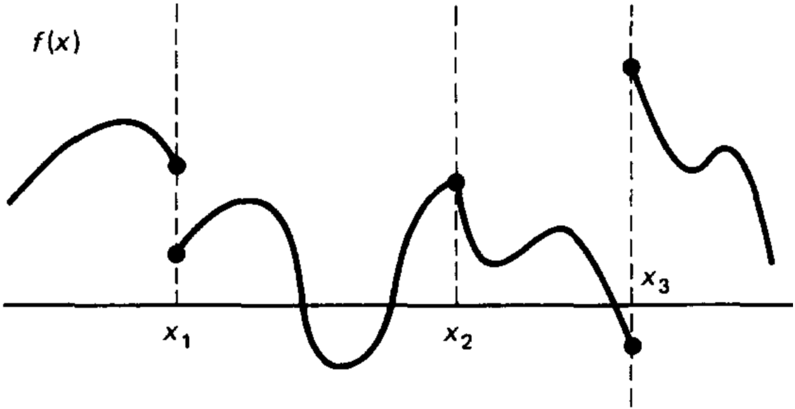
$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

- Complete answers are never easy (and hardly possible) when dealing with the Fourier series!
- A Fourier series converges everywhere provided that $a_n \to 0$ and $b_n \to 0$ fast enough (however fast decay is not necessary).
- The Fourier series of a continuous function converges to this function **almost everywhere**.
- The Fourier series associated to a function converges everywhere provided that the function is **piecewise smooth** (condition may be relaxed).

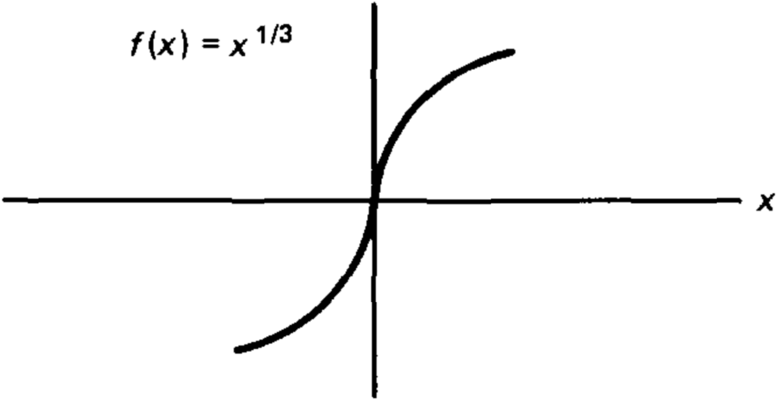


Jump discontinuity

Piecewise continuous = finitely many jump discontinuities



Piecewise smooth function
(both function and its derivative
are piecewise continuous)



Continuous, but not piecewise smooth function

Pointwise convergence

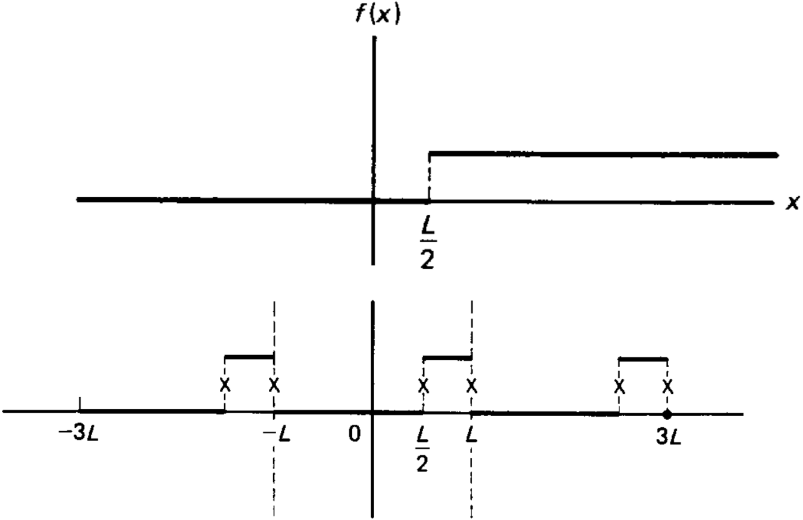
Theorem Suppose $F : [-\pi, \pi] \to \mathbb{R}$ is a piecewise smooth function. Then the Fourier series of F converges everywhere.

The sum at a point x $(-\pi < x < \pi)$ is equal to $F(x)$ if F is continuous at x. Otherwise the sum is equal to

$$\frac{F(x-) + F(x+)}{2}.$$

The sum at the points π and $-\pi$ is equal to

$$\frac{F(\pi-) + F(-\pi+)}{2}.$$



Function and its Fourier series ($L = \pi$)

Example. Fourier series of the function $F(x) = x$.

$$x \sim 2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}$$

$$= 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots \right)$$

The series converges to the function $F(x)$ for any $-\pi < x < \pi$.

For $x = \pi/2$ we obtain:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

Example. Fourier series of the function $f(x) = x^2$.

Proposition Fourier series of an odd function contains only sines, while Fourier series of an even function contains only cosines and a constant term.

Theorem Suppose that a function $f : [-\pi, \pi] \to \mathbb{R}$ is continuous, piecewise smooth, and $f(-\pi) = f(\pi)$.

Then the Fourier series of f' can be obtained via **term-by-term differentiation** of the Fourier series of f.

Example. Fourier series of the function $f(x) = x^2$.

$$x^2 \sim a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$

Term-by-term differentiation yields

$$-a_1 \sin x - 2a_2 \sin 2x - 3a_3 \sin 3x - 4a_4 \sin 4x - \cdots$$

This should be the Fourier series of $f'(x) = 2x$, which is

$$2x \sim 4\left(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \cdots\right).$$

Hence $a_n = (-1)^n \frac{4}{n^2}$ for $n \geq 1$.

It remains to find $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$.

Example. Fourier series of the function $f(x) = x^2$.

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

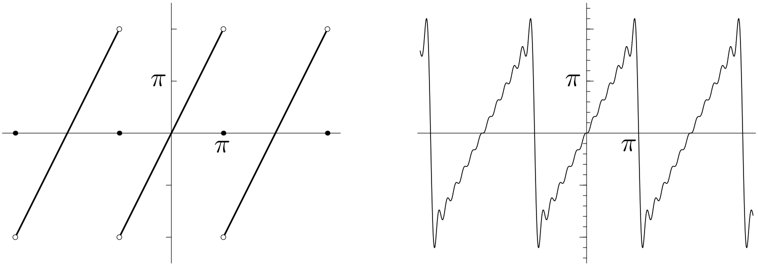
$$= \frac{\pi^2}{3} + 4\left(-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \cdots\right)$$

The series converges to $f(x)$ for any $-\pi \le x \le \pi$.

For $x = 0$ we obtain: $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots$

For $x = \pi$ we obtain: $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$

Gibbs' phenomenon



Left graph: Fourier series of $F(x) = 2x$.
Right graph: 12th partial sum of the series.

The maximal value of the nth partial sum for large n is about 17.9% higher than the maximal value of the series. This is the so-called **Gibbs' overshoot**.