MATH 311-504

Topics in Applied Mathematics

Lecture 3-1: Complex numbers. Complex eigenvalues.

Diagonalization

Let L be a linear operator on a finite-dimensional vector space V. Then the following conditions are equivalent:

- the matrix of L with respect to some basis is diagonal;
- there exists a basis for V formed by eigenvectors of L.

The operator L is **diagonalizable** if it satisfies these conditions.

Let A be an $n \times n$ matrix. Then the following conditions are equivalent:

- A is the matrix of a diagonalizable operator;
- $A = UBU^{-1}$, where B is a diagonal matrix;
- there exists a basis for $\mathbb{R}^n$ formed by eigenvectors of A.

The matrix A is **diagonalizable** if it satisfies these conditions.

There are **two obstructions** to diagonalization of a matrix (i.e., existence of a basis of eigenvectors).

Example 1. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1$.

$\Longrightarrow$ no real eigenvalues or eigenvectors

(However there are *complex* eigenvalues/eigenvectors.)

Example 2. $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

$\det(A - \lambda I) = (\lambda - 1)^2$. Hence $\lambda = 1$ is the only eigenvalue. The associated eigenspace is the line $t(1, 0)$.

Evolution of numbers

Natural numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$.

$$x + 5 = 3, \ \ x = ?$$

Integers: $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

$$7x = 5, \ \ x = ?$$

Rational numbers: $\mathbb{Q} = \{\frac{m}{n} : m, n \in \mathbb{Z}, \ n \neq 0\}$.

$$x^2 = 2, \ \ x = ?$$

Real numbers: $\mathbb{R}$.

$$x^2 + 1 = 0, \ \ x = ?$$

Complex numbers

$\mathbb{C}$: complex numbers.

Complex number: $\boxed{z = x + iy,}$

where $x, y \in \mathbb{R}$ and $i^2 = -1$.

$i = \sqrt{-1}$: imaginary unit

Alternative notation: $z = x + yi$.

$x =$ real part of z,
$iy =$ imaginary part of z

$y = 0 \implies z = x$ (real number)
$x = 0 \implies z = iy$ (purely imaginary number)

We add and multiply complex numbers as polynomials in i (but keep in mind that $i^2 = -1$).

If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

Examples. • $(1+i) - (3+5i) = (1-3) + (i-5i)$ $= -2 - 4i$;

• $(1+i)(3+5i) = 1 \cdot 3 + i \cdot 3 + 1 \cdot 5i + i \cdot 5i$ $= 3 + 3i + 5i + 5i^2 = 3 + 3i + 5i - 5 = -2 + 8i$;

• $2i(3-2i) = 6i - 4i^2 = 4 + 6i$;

• $(2+3i)(2-3i) = 4 - 9i^2 = 4 + 9 = 13$;

• $i^3 = -i, \;\; i^4 = 1, \;\; i^5 = i.$

Let $z = \cos\alpha + i\sin\alpha$, $w = \cos\beta + i\sin\beta$, where $\alpha, \beta \in \mathbb{R}$.

$zw = \cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta + i^2\sin\alpha\sin\beta = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \sin\beta\cos\alpha) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$.

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \\ = \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

As a consequence,

$$(\cos\alpha + i\sin\alpha)^n = \cos(n\alpha) + i\sin(n\alpha).$$

Complex exponentials

Definition. For any $z \in \mathbb{C}$ let

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!} + \cdots$$

Remark. A sequence of complex numbers $z_1 = x_1 + iy_1,\ z_2 = x_2 + iy_2, \ldots$ converges to $z = x + iy$ if $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

Theorem 1 If $z = x + iy$, $x, y \in \mathbb{R}$, then

$$e^z = e^x(\cos y + i \sin y).$$

In particular, $e^{i\phi} = \cos\phi + i\sin\phi,\ \ \phi \in \mathbb{R}$.

Theorem 2 $e^{z+w} = e^z \cdot e^w$ for all $z, w \in \mathbb{C}$.

Proposition $e^{i\phi} = \cos\phi + i\sin\phi$ for all $\phi \in \mathbb{R}$.

Proof: $e^{i\phi} = 1 + i\phi + \frac{(i\phi)^2}{2!} + \cdots + \frac{(i\phi)^n}{n!} + \cdots$

The sequence $1, i, i^2, i^3, \ldots, i^n, \ldots$ is periodic:
$\underbrace{1, i, -1, -i}, \underbrace{1, i, -1, -i}, \ldots$

It follows that

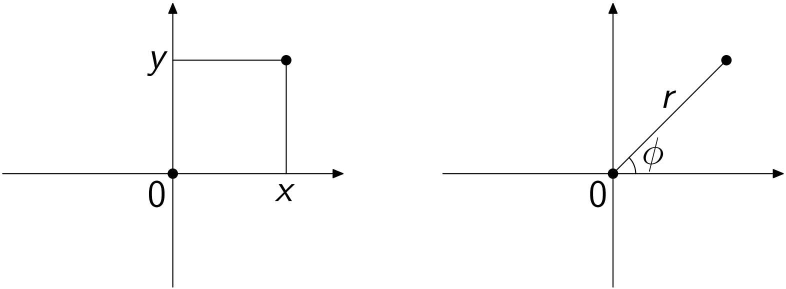
$$e^{i\phi} = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \cdots + (-1)^k \frac{\phi^{2k}}{(2k)!} + \cdots$$

$$+ \; i\left(\phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \cdots + (-1)^k \frac{\phi^{2k+1}}{(2k+1)!} + \cdots\right)$$

$$= \cos\phi + i\sin\phi.$$

Geometric representation

Any complex number $z = x + iy$ is represented by the vector/point $(x, y) \in \mathbb{R}^2$.



$x = r\cos\phi, \quad y = r\sin\phi$

$\implies z = r(\cos\phi + i\sin\phi) = re^{i\phi}$.

$$\boxed{z = r(\cos\phi + i\sin\phi) = re^{i\phi}}$$

$r \geq 0$ is the **modulus** of z (denoted $|z|$).
$|x + iy| = \sqrt{x^2 + y^2}$.

$\phi \in \mathbb{R}$ is the **argument** of z (determined up to adding a multiple of 2π).

$$z_1 = r_1 e^{i\phi_1}, \;\; z_2 = r_2 e^{i\phi_2} \quad \Longrightarrow \quad z_1 z_2 = r_1 r_2 e^{i(\phi_1 + \phi_2)}$$

Division

If $z = re^{i\phi}$, then $z^{-1} = r^{-1}e^{-i\phi}$ because $re^{i\phi} \cdot r^{-1}e^{-i\phi} = rr^{-1}e^{i(\phi-\phi)} = e^{i0} = 1.$

$$z_1 = r_1 e^{i\phi_1}, \;\; z_2 = r_2 e^{i\phi_2} \quad \Longrightarrow \quad \frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\phi_1 - \phi_2)}$$

Given $z = x + iy$, the **complex conjugate** of z is $\bar{z} = x - iy$. The conjugacy $z \mapsto \bar{z}$ is the reflection of $\mathbb{C}$ in the real line.

$$z\bar{z} = (x + iy)(x - iy) = x^2 - (iy)^2 = x^2 + y^2 = |z|^2.$$

$$z^{-1} = \frac{\bar{z}}{|z|^2}, \qquad (x + iy)^{-1} = \frac{x - iy}{x^2 + y^2}.$$

Examples.

- $i^{-1} = \dfrac{i}{i^2} = -i;$

- $$\frac{1}{1-i} = \frac{1+i}{(1-i)(1+i)} = \frac{1+i}{1-i^2} = \frac{1}{2} + \frac{1}{2}i;$$

- $$\frac{2+3i}{1+2i} = \frac{(2+3i)(1-2i)}{(1+2i)(1-2i)} = \frac{8-i}{5} = 1.6 - 0.2i.$$

Roots of unity

Problem. Solve the equation $z^n - 1 = 0$ $(n \geq 1)$.

Let $z = re^{i\phi}$ $(r > 0,\ \phi \in \mathbb{R})$. Then $z^n = r^n e^{in\phi}$.
Hence $z^n = 1$ if $r^n = 1$ and $n\phi = 2\pi k,\ k \in \mathbb{Z}$.
That is, $r = 1,\ \phi = 2\pi k/n,\ k \in \mathbb{Z}$.

Solutions: $z_k = \cos \dfrac{2\pi k}{n} + i \sin \dfrac{2\pi k}{n},\ \ 0 \leq k \leq n-1.$

Cubic roots of unity: $1,\ -\frac{1}{2} + \frac{\sqrt{3}}{2}i,\ -\frac{1}{2} - \frac{\sqrt{3}}{2}i.$

Roots of unity of degree 4: $1,\ i,\ -1,\ -i.$

Fundamental Theorem of Algebra
Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Complex eigenvalues/eigenvectors

Example. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$.
Characteristic values: $\lambda_1 = i$ and $\lambda_2 = -i$.
Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$