MATH 311-504

Topics in Applied Mathematics

Lecture 3-2: Complex eigenvalues and eigenvectors. Norm.

Fundamental Theorem of Algebra
Any polynomial of degree $n \geq 1$, with complex coefficients, has exactly n roots (counting with multiplicities).

Equivalently, if

$$p(z) = a_n z^n + a_{n-1} z + \cdots + a_1 z + a_0,$$

where $a_i \in \mathbb{C}$ and $a_n \neq 0$, then there exist complex numbers $z_1, z_2, \ldots, z_n$ such that

$$p(z) = a_n (z - z_1)(z - z_2) \ldots (z - z_n).$$

Complex eigenvalues/eigenvectors

Example. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$.
Characteristic values: $\lambda_1 = i$ and $\lambda_2 = -i$.
Associated eigenvectors: $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$.

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} i \\ 1 \end{pmatrix} = i \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} -i \\ 1 \end{pmatrix} = -i \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

$\mathbf{v}_1$, $\mathbf{v}_2$ is a basis of eigenvectors. *In which space?*

Complexification

Instead of the real vector space $\mathbb{R}^2$, we consider a complex vector space $\mathbb{C}^2$ (all complex numbers are admissible as scalars).

The linear operator $f : \mathbb{R}^2 \to \mathbb{R}^2$, $f(\mathbf{x}) = A\mathbf{x}$ is replaced by the complexified linear operator $F : \mathbb{C}^2 \to \mathbb{C}^2$, $F(\mathbf{x}) = A\mathbf{x}$.

The vectors $\mathbf{v}_1 = (1, -i)$ and $\mathbf{v}_2 = (1, i)$ form a basis for $\mathbb{C}^2$.

$$\begin{vmatrix} 1 & 1 \\ -i & i \end{vmatrix} = 2i \neq 0.$$

Example. $A_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$.

Linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$, $L(\mathbf{x}) = A_\phi \mathbf{x}$ is the rotation about the origin by the angle ϕ (counterclockwise).

Characteristic equation: $\begin{vmatrix} \cos\phi - \lambda & -\sin\phi \\ \sin\phi & \cos\phi - \lambda \end{vmatrix} = 0.$

$$(\cos\phi - \lambda)^2 + \sin^2\phi = 0.$$

$\lambda_1 = \cos\phi + i\sin\phi = e^{i\phi}$, $\lambda_2 = \cos\phi - i\sin\phi = e^{-i\phi}$.

Consider vectors $\mathbf{v}_1 = (1, -i)$, $\mathbf{v}_2 = (1, i)$.

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} \cos\phi + i\sin\phi \\ \sin\phi - i\cos\phi \end{pmatrix} = e^{i\phi} \begin{pmatrix} 1 \\ -i \end{pmatrix},$$

$$\begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} 1 \\ i \end{pmatrix} = \begin{pmatrix} \cos\phi - i\sin\phi \\ \sin\phi + i\cos\phi \end{pmatrix} = e^{-i\phi} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Thus $A_\phi \mathbf{v}_1 = e^{i\phi}\mathbf{v}_1$, $A_\phi \mathbf{v}_2 = e^{-i\phi}\mathbf{v}_2$.

Beyond linear structure

n-dimensional coordinate vector space $\mathbb{R}^n$ carries additional structure: *length* and *dot product*.

Let $\mathbf{x} = (x_1, x_2, \dots, x_n)$, $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$.

Length: $|\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

Dot product: $\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

Length and dot product $\implies$ angle between vectors

Angle: $\angle(\mathbf{x}, \mathbf{y}) = \arccos \dfrac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|\,|\mathbf{y}|}$.

Orthogonality: $\angle(\mathbf{x}, \mathbf{y}) = 90^\circ$ if $\mathbf{x} \cdot \mathbf{y} = 0$.

Properties of the length function:

(i) $|\mathbf{x}| \geq 0$, $|\mathbf{x}| = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $|r\mathbf{x}| = |r|\,|\mathbf{x}|$ for all $r \in \mathbb{R}$ (homogeneity)

(iii) $|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}|$ (triangle inequality)

Properties of the dot product:

(i) $\mathbf{x} \cdot \mathbf{x} \geq 0$, $\mathbf{x} \cdot \mathbf{x} = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)

(ii) $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$ (symmetry)

(iii) $(r\mathbf{x}) \cdot \mathbf{y} = r(\mathbf{x} \cdot \mathbf{y})$ (homogeneity)

(iv) $(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}$ (distributive law)

(iii) and (iv) $\implies$ $\mathbf{x} \cdot \mathbf{y}$ is a linear function of $\mathbf{x}$

(ii) $\implies$ $\mathbf{x} \cdot \mathbf{y}$ is a linear function of $\mathbf{y}$ as well

That is, the dot product is a *bilinear* function.

Relation between length and dot product: $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$

Norm

The notion of *norm* generalizes the notion of length of a vector in $\mathbb{R}^n$.

Definition. Let V be a vector space. A function $\alpha : V \to \mathbb{R}$ is called a **norm** on V if it has the following properties:

(i) $\alpha(\mathbf{x}) \geq 0$, $\alpha(\mathbf{x}) = 0$ only for $\mathbf{x} = \mathbf{0}$ (positivity)
(ii) $\alpha(r\mathbf{x}) = |r|\,\alpha(\mathbf{x})$ for all $r \in \mathbb{R}$ (homogeneity)
(iii) $\alpha(\mathbf{x} + \mathbf{y}) \leq \alpha(\mathbf{x}) + \alpha(\mathbf{y})$ (triangle inequality)

Notation. The norm of a vector $\mathbf{x} \in V$ is usually denoted $\|\mathbf{x}\|$. Different norms on V are distinguished by subscripts, e.g., $\|\mathbf{x}\|_1$ and $\|\mathbf{x}\|_2$.

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_\infty = \max(|x_1|, |x_2|, \ldots, |x_n|)$.

Positivity and homogeneity are obvious.
The triangle inequality:

$$|x_i + y_i| \le |x_i| + |y_i| \le \max_j |x_j| + \max_j |y_j|$$
$$\implies \max_j |x_j + y_j| \le \max_j |x_j| + \max_j |y_j|$$

- $\|\mathbf{x}\|_1 = |x_1| + |x_2| + \cdots + |x_n|$.

Positivity and homogeneity are obvious.
The triangle inequality: $|x_i + y_i| \le |x_i| + |y_i|$

$$\implies \sum_j |x_j + y_j| \le \sum_j |x_j| + \sum_j |y_j|$$

Examples. $V = \mathbb{R}^n$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$.

- $\|\mathbf{x}\|_p = \big(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\big)^{1/p}$, $p > 0$.

Theorem $\|\mathbf{x}\|_p$ is a norm on $\mathbb{R}^n$ for any $p \geq 1$.

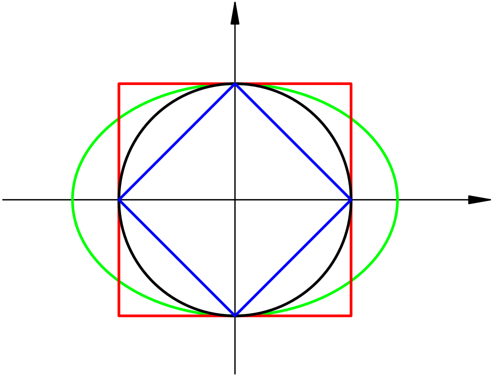
Remark. $\|\mathbf{x}\|_2 = |\mathbf{x}|$.

Definition. A **normed vector space** is a vector space endowed with a norm.

The norm defines a distance function on the normed vector space: $\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$.

Then we say that a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ converges to a vector $\mathbf{x}$ if $\mathrm{dist}(\mathbf{x}, \mathbf{x}_n) \to 0$ as $n \to \infty$.

Unit circle: $\|\mathbf{x}\| = 1$



$\|\mathbf{x}\| = (x_1^2 + x_2^2)^{1/2}$	black
$\|\mathbf{x}\| = \left(\frac{1}{2}x_1^2 + x_2^2\right)^{1/2}$	green
$\|\mathbf{x}\| = \|x_1\| + \|x_2\|$	blue
$\|\mathbf{x}\| = \max(\|x_1\|, \|x_2\|)$	red

Examples. $V = C[a,b]$, $f : [a,b] \to \mathbb{R}$.

- $\|f\|_\infty = \max_{a \le x \le b} |f(x)|$ (uniform norm).

- $\|f\|_1 = \int_a^b |f(x)|\, dx$.

- $\|f\|_p = \left(\int_a^b |f(x)|^p\, dx \right)^{1/p}$, $p > 0$.

Theorem $\|f\|_p$ is a norm on $C[a,b]$ for any $p \ge 1$.