Math 311-504

Topics in Applied Mathematics

Lecture 3:
Lines and planes (continued).
Systems of linear equations.

Lines

Definition. A *line* in $\mathbb{R}^n$ is a set of all points $t\mathbf{u} + \mathbf{v}$, where $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v}$ are fixed vectors in $\mathbb{R}^n$ while t ranges over all real numbers.

$t\mathbf{u} + \mathbf{v}$ is a *parametric representation* of the line.

In $\mathbb{R}^2$, a line is given by an equation $ax + by = c$, where a, b, c are constants. The vector (a, b) is orthogonal to the line.

Planes

Definition. A *plane* in $\mathbb{R}^n$ is a set of all points $t\mathbf{u} + s\mathbf{w} + \mathbf{v}$, where $\mathbf{u}$, $\mathbf{w}$, and $\mathbf{v}$ are fixed vectors in $\mathbb{R}^n$ such that $\mathbf{u}$ and $\mathbf{w}$ are not parallel, while t and s range over all real numbers.

$t\mathbf{u} + s\mathbf{w} + \mathbf{v}$ is a *parametric representation*.

In $\mathbb{R}^3$, a plane is given by an equation

$$ax + by + cz = d,$$

where a, b, c, d are constants. The vector (a, b, c) is orthogonal to the plane.

Problem 1. Determine whether the point $(1,4)$ lies on the line $t(2,-1)+(-3,5)$.

We need to check solvability of the equation $t(2,-1)+(-3,5)=(1,4)$. This vector equation is equivalent to a system of two scalar equations:

$$\begin{cases} 2t-3=1 \\ -t+5=4 \end{cases} \iff \begin{cases} t=2 \\ t=1 \end{cases}$$

The system has no solution $\implies$ the point is not on the line.

Problem 2. Find an equation for the line $t(2,-1)+(-3,5)$.

The vector (a,b) is orthogonal to $(-b,a)$ for any $a,b\in\mathbb{R}$. Hence the vector $(1,2)$ is orthogonal to the line. Therefore the equation is $1(x+3)+2(y-5)=0$ or $x+2y=7$.

Problem 3. Find a parametric representation for a plane in $\mathbb{R}^3$ given by the equation $3x + 2y + z = 5$.

The equation can be solved for z: $z = 5 - 3x - 2y$. It follows that the plane has a parametric representation

$$\begin{cases} x = t, \\ y = s, \\ z = 5 - 3t - 2s, \end{cases} \qquad t, s \in \mathbb{R}.$$

That is, $(x, y, z) = (t, s, 5 - 3t - 2s) =$
$= t(1, 0, -3) + s(0, 1, -2) + (0, 0, 5)$.

Problem 4. Find a parametric representation and an equation for the plane passing through $(1, 1, 1)$ and parallel to the plane $3x + 2y + z = 5$.

Parametric representation: $t(1, 0, -3) + s(0, 1, -2) + (1, 1, 1)$.
Equation: $3x + 2y + z = d$, where d is a constant. Since the point $(1, 1, 1)$ is on the plane, we get $d = 3 + 2 + 1 = 6$.

Problem 5. Find the intersection point of the line $t(2,-1,0)+(-3,5,0)$ with the plane $3x+2y+z=5$.

An arbitrary point on the line has coordinates

$$(x,y,z)=t(2,-1,0)+(-3,5,0)=(2t-3,-t+5,0),$$

where $t\in\mathbb{R}$. The point belongs to the plane if and only if

$$3(2t-3)+2(-t+5)+0=5$$
$$\iff 4t+1=5 \iff t=1.$$

Thus the intersection point is

$$1(2,-1,0)+(-3,5,0)=(-1,4,0).$$

Linear equations

An equation

$$\boxed{ax + by = c,}$$

where x, y are variables and a, b, c are constants, is called **linear** (because it defines a line in $\mathbb{R}^2$).

More generally, a **linear equation** is any equation of the form

$$\boxed{a_1x_1 + a_2x_2 + \cdots + a_nx_n = b,}$$

where $x_1, x_2, \ldots, x_n$ are variables and $a_1, a_2, \ldots, a_n, b$ are constants.

Systems of linear equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Here $x_1, x_2, \ldots, x_n$ are variables and a_{ij}, b_j are constants.

A *solution* of the system is a common solution of all equations in the system.

Plenty of problems in mathematics and applications require solving systems of linear equations.

Problem. Find the point of intersection of the lines $x - y = -2$ and $2x + 3y = 6$ in $\mathbb{R}^2$.

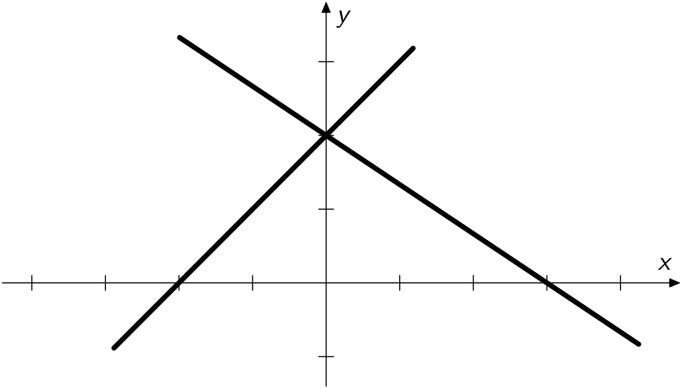
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 2x + 3y = 6 \end{cases} \iff$$

$$\begin{cases} x = y - 2 \\ 2(y-2) + 3y = 6 \end{cases} \iff \begin{cases} x = y - 2 \\ 5y = 10 \end{cases} \iff$$

$$\begin{cases} x = y - 2 \\ y = 2 \end{cases} \iff \begin{cases} x = 0 \\ y = 2 \end{cases}$$

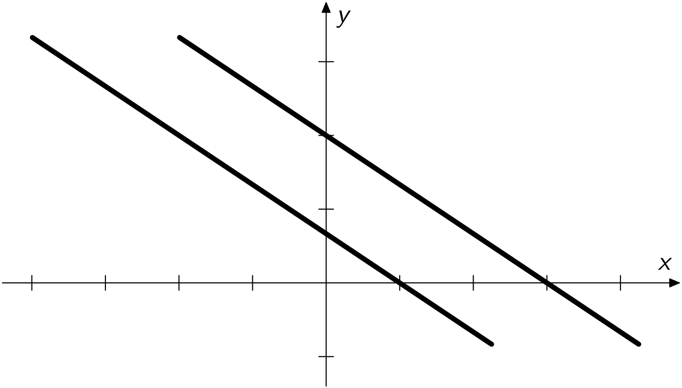
Solution: the lines intersect at the point $(0, 2)$.

Remark. The symbol of equivalence $\iff$ means that two systems have the same solutions.



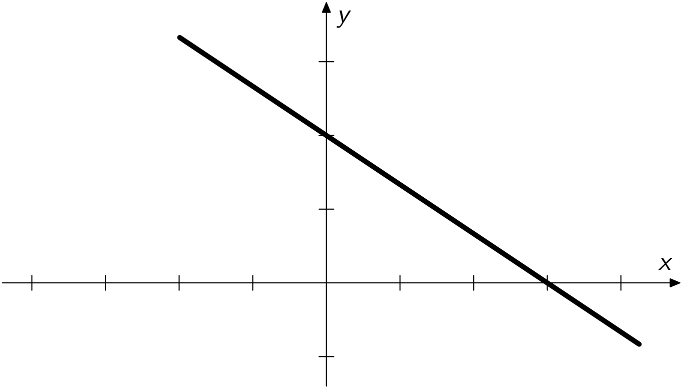
$$\begin{cases} x - y = -2 \\ 2x + 3y = 6 \end{cases}$$

$$x = 0,\ y = 2$$



$$\begin{cases} 2x + 3y = 2 \\ 2x + 3y = 6 \end{cases}$$

inconsistent system
(no solutions)



$$\begin{cases} 4x + 6y = 12 \\ 2x + 3y = 6 \end{cases} \iff 2x + 3y = 6$$

Solving systems of linear equations

Elimination method always works for systems of linear equations.

Algorithm: (1) pick a variable, solve one of the equations for it, and eliminate it from the other equations; (2) put aside the equation used in the elimination, and return to step (1).

The algorithm reduces the number of variables (as well as the number of equations), hence it stops after a finite number of steps.

After the algorithm stops, the system is simplified so that it should be clear how to complete solution.

Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solve the 1st equation for x:

$$\begin{cases} x = y + 2 \\ 2x - y - z = 3 \\ x + y + z = 6 \end{cases}$$

Eliminate x from the 2nd and 3rd equations:

$$\begin{cases} x = y + 2 \\ 2(y + 2) - y - z = 3 \\ (y + 2) + y + z = 6 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y - z = -1 \\ 2y + z = 4 \end{cases}$$

Now the 2nd and 3rd equations form a system of two linear equations in two variables.

Solve the 2nd equation for y, then eliminate y from the 3rd equation:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 2y + z = 4 \end{cases} \qquad \begin{cases} x = y + 2 \\ y = z - 1 \\ 2(z - 1) + z = 4 \end{cases}$$

Simplify:

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ 3z = 6 \end{cases}$$

The elimination is completed. Now the system is easily solved by back substitution.

That is, we find z from the 3rd equation, then substitute it in the 2nd equation and find y, then substitute y and z in the 1st equation and find x.

$$\begin{cases} x = y + 2 \\ y = z - 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = y + 2 \\ y = 1 \\ z = 2 \end{cases} \qquad \begin{cases} x = 3 \\ y = 1 \\ z = 2 \end{cases}$$

System of linear equations:

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases}$$

Solution: $(x, y, z) = (3, 1, 2)$