

Math 311-504

Topics in Applied Mathematics

**Lecture 6:**

**Row echelon form (continued).**

**Linear independence.**

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Coefficient matrix ( $m \times n$ ) and column vector of the right-hand sides ( $m \times 1$ ):

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

*Augmented  $m \times (n + 1)$  matrix:*

$$\left( \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right)$$

Solution of a system of linear equations splits into two parts: **(A)** elimination and **(B)** back substitution.

Both parts can be done by applying a finite number of **elementary operations**.

Since the elementary operations preserve the standard form of linear equations, we can trace the solution process by looking on the **augmented matrix**.

In terms of the augmented matrix, the elementary operations are **elementary row operations**.

**Elementary operations** for systems of linear equations:

- (1) to multiply an equation by a nonzero scalar;
- (2) to add an equation multiplied by a scalar to another equation;
- (3) to interchange two equations.

**Elementary row operations:**

- (1) to multiply a row by some  $r \neq 0$ ;
- (2) to add a row multiplied by some  $r \in \mathbb{R}$  to another row;
- (3) to interchange two rows.

*Remark.* The rows are added and multiplied by scalars as vectors (namely, **row vectors**).

$$\left( \begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array} \right) = \left( \begin{array}{c} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{array} \right),$$

where  $\mathbf{v}_i = (a_{i1} \ a_{i2} \ \dots \ a_{in} \mid b_i)$  is a row vector.

*Operation 1:* to multiply the  $i$ th row by  $r \neq 0$ :

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

*Operation 2:* to add the  $i$ th row multiplied by  $r$  to the  $j$ th row:

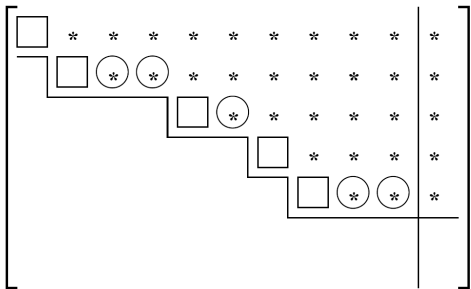
$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j + r\mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$



*Operation 3:* to interchange the  $i$ th row with the  $j$ th row:

$$\begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \rightarrow \begin{pmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_j \\ \vdots \\ \mathbf{v}_i \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

The goal of the Gaussian elimination is to convert the augmented matrix into **row echelon form**:



- all the entries below the staircase line are zero;
- boxed entries, called **pivot** or **leading entries**, are nonzero;
- each circled star corresponds to a free variable.

**Strict triangular form** is a particular case of row echelon form that can occur for systems of  $n$  equations in  $n$  variables:

$$\begin{bmatrix} \square & * & * & * & * & * & * \\ & \square & * & * & * & * & * \\ & & \square & * & * & * & * \\ & & & \square & * & * & * \\ & & & & \square & * & * \\ & & & & & \square & * \\ & & & & & & \square \end{bmatrix}$$

Matrix of  
coefficients

The original system of linear equations is **consistent** if there is no leading entry in the rightmost column of the augmented matrix in row echelon form.

The diagram shows an augmented matrix in row echelon form. The matrix is enclosed in large square brackets. A vertical line separates the coefficient matrix from the rightmost column. The matrix has 6 rows and 11 columns. The leading entries are squares in the first column of each row, forming a staircase pattern. The rightmost column contains asterisks. The second, fourth, and sixth rows have two asterisks circled in the second, fourth, and sixth columns, respectively, indicating that these rows have a leading entry in the rightmost column.

Inconsistent system

The goal of the back substitution is to go from row echelon form to **reduced row echelon form** (or simply **reduced form**):

$$\left[ \begin{array}{cccccccccccc|c} \boxed{1} & 0 & * & * & 0 & * & 0 & 0 & * & * & * & * \\ & \boxed{1} & \circledast & \circledast & 0 & * & 0 & 0 & * & * & * & * \\ & & & & \boxed{1} & \circledast & 0 & 0 & * & * & * & * \\ & & & & & & \boxed{1} & 0 & * & * & * & * \\ & & & & & & & \boxed{1} & \circledast & \circledast & * & * \\ & & & & & & & & & & & * \end{array} \right]$$

- all entries below the staircase line are zero;
- each leading entry is 1, the other entries in its column are zero;
- each circled star corresponds to a free variable.

### Example.

$$\begin{cases} x - y & = 2 \\ 2x - y - z & = 3 \\ x + y + z & = 6 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & -1 & 0 & 2 \\ 2 & -1 & -1 & 3 \\ 1 & 1 & 1 & 6 \end{array} \right)$$

Row echelon form (also strict triangular):

$$\begin{cases} x - y & = 2 \\ y - z & = -1 \\ 3z & = 6 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & -1 & 0 & 2 \\ 0 & \boxed{1} & -1 & -1 \\ 0 & 0 & \boxed{3} & 6 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x & = 3 \\ y & = 1 \\ z & = 2 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 1 \\ 0 & 0 & \boxed{1} & 2 \end{array} \right)$$

## Another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 1 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 1 \end{array} \right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = -13 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & \boxed{-13} \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x - z = 0 \\ y - z = 0 \\ 0 = 1 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 0 & -1 & 0 \\ 0 & \boxed{1} & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{array} \right)$$

## Yet another example.

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ -x + 4y - 3z = 14 \end{cases} \quad \left( \begin{array}{ccc|c} 1 & 1 & -2 & 1 \\ 0 & 1 & -1 & 3 \\ -1 & 4 & -3 & 14 \end{array} \right)$$

Row echelon form:

$$\begin{cases} x + y - 2z = 1 \\ y - z = 3 \\ 0 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 1 & -2 & 1 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

Reduced row echelon form:

$$\begin{cases} x - z = -2 \\ y - z = 3 \\ 0 = 0 \end{cases} \quad \left( \begin{array}{ccc|c} \boxed{1} & 0 & -1 & -2 \\ 0 & \boxed{1} & -1 & 3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$



**New example.** 
$$\begin{cases} x_2 + 2x_3 + 3x_4 = 6 \\ x_1 + 2x_2 + 3x_3 + 4x_4 = 10 \end{cases}$$

Augmented matrix: 
$$\left( \begin{array}{cccc|c} 0 & 1 & 2 & 3 & 6 \\ 1 & 2 & 3 & 4 & 10 \end{array} \right)$$

To obtain row echelon form, interchange the rows:

$$\left( \begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 10 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array} \right)$$

*The system is consistent. There are two free variables:  $x_3$  and  $x_4$ .*

To obtain reduced row echelon form, add  $-2$  times the 2nd row to the 1st row:

$$\left( \begin{array}{cccc|c} \boxed{1} & 2 & 3 & 4 & 10 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array} \right) \rightarrow \left( \begin{array}{cccc|c} \boxed{1} & 0 & -1 & -2 & -2 \\ 0 & \boxed{1} & 2 & 3 & 6 \end{array} \right)$$

$$\begin{cases} x_1 - x_3 - 2x_4 = -2 \\ x_2 + 2x_3 + 3x_4 = 6 \end{cases} \iff \begin{cases} x_1 = x_3 + 2x_4 - 2 \\ x_2 = -2x_3 - 3x_4 + 6 \end{cases}$$

**General solution:**

$$\begin{cases} x_1 = t + 2s - 2 \\ x_2 = -2t - 3s + 6 \\ x_3 = t \\ x_4 = s \end{cases} \quad (t, s \in \mathbb{R})$$

$$\begin{aligned} (x_1, x_2, x_3, x_4) &= (t + 2s - 2, -2t - 3s + 6, t, s) = \\ &= t(1, -2, 1, 0) + s(2, -3, 0, 1) + (-2, 6, 0, 0). \end{aligned}$$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The system is consistent if there is no leading entry in the rightmost column of the reduced augmented matrix. In this case, the general solution is

$$(x_1, x_2, \dots, x_n) = t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \cdots + t_k \mathbf{v}_k + \mathbf{v}_0,$$

where  $\mathbf{v}_i$  are certain  $n$ -dimensional (row) vectors and  $t_i$  are arbitrary scalars.

$$k = n - (\# \text{ of leading entries})$$

*Definition.* A subset  $S \subset \mathbb{R}^n$  is called a **hyperplane** (or an **affine subspace**) if it has a parametric representation  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k + \mathbf{v}_0$ , where  $\mathbf{v}_i$  are fixed  $n$ -dimensional vectors and  $t_i$  are arbitrary scalars.

Hyperplanes are solution sets of systems of linear equations.

The number  $k$  of parameters may depend on a representation. The hyperplane  $S$  is called a  **$k$ -plane** if  $k$  is as small as possible.

*Example.* Suppose  $\mathbf{v}_2 = r\mathbf{v}_1$ , where  $r \in \mathbb{R}$ .

Then  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \mathbf{v}_0 = (t_1 + rt_2)\mathbf{v}_1 + \mathbf{v}_0$ .

Hence  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \mathbf{v}_0$  and  $t\mathbf{v}_1 + \mathbf{v}_0$  are different representations of the same hyperplane.

0-plane is a point.

$t\mathbf{v}_1 + \mathbf{v}_0$  is a 1-plane if  $\mathbf{v}_1 \neq \mathbf{0}$ .

$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \mathbf{v}_0$  is a 2-plane if vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are not parallel.

Thus 1-planes are lines, 2-planes are planes.

*Definition.* Given vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  and scalars  $t_1, t_2, \dots, t_k$ , the vector

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k$$

is called a **linear combination** of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$ . The linear combination is called **nontrivial** if the coefficients  $t_1, \dots, t_k$  are not all equal to zero.

**Proposition** The following conditions are equivalent:

- (i) the zero vector is a nontrivial linear combination of  $\mathbf{v}_1, \dots, \mathbf{v}_k$ ;
- (ii) one of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k$  is a linear combination of the other  $k - 1$  vectors.

*Proof:* (i)  $\implies$  (ii) Suppose that

$$t_1 \mathbf{v}_1 + t_2 \mathbf{v}_2 + \cdots + t_k \mathbf{v}_k = \mathbf{0},$$

where  $t_i \neq 0$  for some  $1 \leq i \leq k$ . Then

$$\mathbf{v}_i = -\frac{t_1}{t_i} \mathbf{v}_1 - \cdots - \frac{t_{i-1}}{t_i} \mathbf{v}_{i-1} - \frac{t_{i+1}}{t_i} \mathbf{v}_{i+1} - \cdots - \frac{t_k}{t_i} \mathbf{v}_k.$$

(ii)  $\implies$  (i) Suppose that

$$\mathbf{v}_i = r_1 \mathbf{v}_1 + \cdots + r_{i-1} \mathbf{v}_{i-1} + r_{i+1} \mathbf{v}_{i+1} + \cdots + r_k \mathbf{v}_k$$

for some scalars  $r_j$ . Then

$$r_1 \mathbf{v}_1 + \cdots + r_{i-1} \mathbf{v}_{i-1} - \mathbf{v}_i + r_{i+1} \mathbf{v}_{i+1} + \cdots + r_k \mathbf{v}_k = \mathbf{0}.$$

*Definition.* Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  are called **linearly dependent** if they satisfy condition (i) or (ii) of the proposition. Otherwise the vectors are called **linearly independent**.

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  are linearly independent if  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_k\mathbf{v}_k = \mathbf{0} \implies t_1 = \dots = t_k = 0$

Let  $\mathbf{v}_i = (a_{1i}, a_{2i}, \dots, a_{ni})$  for  $i = 1, 2, \dots, k$ . Then the vector identity  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \dots + t_k\mathbf{v}_k = \mathbf{0}$  is equivalent to the system

$$\begin{cases} a_{11}t_1 + a_{12}t_2 + \dots + a_{1k}t_k = 0 \\ a_{21}t_1 + a_{22}t_2 + \dots + a_{2k}t_k = 0 \\ \dots\dots\dots \\ a_{n1}t_1 + a_{n2}t_2 + \dots + a_{nk}t_k = 0 \end{cases}$$

Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are columns of the coefficient matrix. The system is consistent. The zero solution is unique if the number of nonzero rows in the reduced matrix is exactly  $k$ .



*Definition.* A system of linear equations is called **homogeneous** if all right-hand sides are zeros.

**Proposition** A homogeneous system can not have a unique solution if the number of equations is less than the number of variables.

**Corollary** Vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k \in \mathbb{R}^n$  are linearly dependent whenever  $k > n$ .

**Theorem** A hyperplane

$$t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + \cdots + t_k\mathbf{v}_k + \mathbf{v}_0$$

is a  $k$ -plane if and only if vectors  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  are linearly independent.

**Problem.** Determine whether vectors  $\mathbf{v}_1 = (1, 2, 1)$ ,  $\mathbf{v}_2 = (-1, -1, 1)$ , and  $\mathbf{v}_3 = (0, -1, 1)$  are linearly dependent.

We need to check if the vector equation  $t_1\mathbf{v}_1 + t_2\mathbf{v}_2 + t_3\mathbf{v}_3 = \mathbf{0}$  has solutions other than  $t_1 = t_2 = t_3 = 0$ .

The vector equation is equivalent to a system of 3 linear equations in variables  $t_1, t_2, t_3$ . Vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are columns of the coefficient matrix of the system.

Augmented matrix: 
$$\left( \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 0 \end{array} \right)$$

Row echelon form: 
$$\left( \begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 3 & 0 \end{array} \right)$$

There are no free variables  $\implies$  the zero solution is unique  
 $\implies$  the vectors are linearly independent