Sample problems for Test 1: Solutions

Any problem may be altered or replaced by a different one!

Problem 1 (25 pts.) Let Π be the plane in $\mathbb{R}^3$ passing through the points $(2,0,0)$, $(1,1,0)$, and $(-3,0,2)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the point $(1,1,1)$ in the direction $(2,2,2)$.

(i) Find a parametric representation for the line ℓ.

$t(2,2,2)+(1,1,1)$. Since the line ℓ passes through the origin ($t=-1/2$), an equivalent representation is $s(2,2,2)$.

(ii) Find a parametric representation for the plane Π.

Since the plane Π contains the points $\mathbf{a}=(2,0,0)$, $\mathbf{b}=(1,1,0)$, and $\mathbf{c}=(-3,0,2)$, the vectors $\mathbf{b}-\mathbf{a}=(-1,1,0)$ and $\mathbf{c}-\mathbf{a}=(-5,0,2)$ are parallel to Π. Clearly, $\mathbf{b}-\mathbf{a}$ is not parallel to $\mathbf{c}-\mathbf{a}$. Hence a parametric representation $t_1(\mathbf{b}-\mathbf{a})+t_2(\mathbf{c}-\mathbf{a})+\mathbf{a}=t_1(-1,1,0)+t_2(-5,0,2)+(2,0,0)$.

(iii) Find an equation for the plane Π.

Since the vectors $\mathbf{b}-\mathbf{a}=(-1,1,0)$ and $\mathbf{c}-\mathbf{a}=(-5,0,2)$ are parallel to the plane Π, their cross product $\mathbf{p}=(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})$ is orthogonal to Π. We have that

$$\mathbf{p}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\-1&1&0\\-5&0&2\end{vmatrix}=\begin{vmatrix}1&0\\0&2\end{vmatrix}\mathbf{i}-\begin{vmatrix}-1&0\\-5&2\end{vmatrix}\mathbf{j}+\begin{vmatrix}-1&1\\-5&0\end{vmatrix}\mathbf{k}=2\mathbf{i}+2\mathbf{j}+5\mathbf{k}=(2,2,5).$$

A point $\mathbf{x}=(x,y,z)$ is in the plane Π if and only if $\mathbf{p}\cdot(\mathbf{x}-\mathbf{a})=0$. This is an equation for the plane. In coordinate form, $2(x-2)+2y+5z=0$ or $2x+2y+5z=4$.

(iv) Find the point where the line ℓ intersects the plane Π.

Let $\mathbf{x}$ be the point of intersection. Then $\mathbf{x}=t_1(-1,1,0)+t_2(-5,0,2)+(2,0,0)$ for some $t_1,t_2\in\mathbb{R}$ and also $\mathbf{x}=s(2,2,2)$ for some $s\in\mathbb{R}$. It follows that

$$\begin{cases}-t_1-5t_2+2=2s,\\ t_1=2s,\\ 2t_2=2s.\end{cases}$$

Solving this system of linear equations, we obtain that $t_1=4/9$, $t_2=s=2/9$. Hence $\mathbf{x}=s(2,2,2)=(4/9,4/9,4/9)$.

(v) Find the angle between the line ℓ and the plane Π.

Let ϕ denote the angle between the vectors $\mathbf{v}=(2,2,2)$ and $\mathbf{p}=(2,2,5)$. Then

$$\cos\phi=\frac{\mathbf{v}\cdot\mathbf{p}}{|\mathbf{v}|\,|\mathbf{p}|}=\frac{2\cdot2+2\cdot2+2\cdot5}{\sqrt{2^2+2^2+2^2}\,\sqrt{2^2+2^2+5^2}}=\frac{18}{\sqrt{12}\,\sqrt{33}}=\frac{3}{\sqrt{11}}.$$

Note that $0 < \phi < \pi/2$ as $\cos\phi > 0$. Since the vector $\mathbf{v}$ is parallel to the line ℓ while the vector $\mathbf{p}$ is orthogonal to the plane Π, the angle between ℓ and Π is equal to

$$\frac{\pi}{2} - \phi = \frac{\pi}{2} - \arccos\frac{3}{\sqrt{11}} = \arcsin\frac{3}{\sqrt{11}}.$$

(vi) Find the distance from the origin to the plane Π.

The plane Π can be defined by the equation $2x + 2y + 5z = 4$. Hence the distance from a point (x_0, y_0, z_0) to Π is equal to

$$\frac{|2x_0 + 2y_0 + 5z_0 - 4|}{\sqrt{2^2 + 2^2 + 5^2}} = \frac{|2x_0 + 2y_0 + 5z_0 - 4|}{\sqrt{33}}.$$

In particular, the distance from the origin to the plane is equal to $\frac{4}{\sqrt{33}}$.

Problem 2 (15 pts.) Let $f(x) = a\cos 2x + b\cos x + c$. Find a, b, and c so that $f(0) = 0$, $f''(0) = 2$, and $f''''(0) = 10$.

$f''(x) = -4a\cos 2x - b\cos x$, $f''''(x) = 16a\cos 2x + b\cos x$. Therefore $f(0) = a + b + c$, $f''(0) = -4a - b$, $f''''(0) = 16a + b$. The desired parameters satisfy a system of linear equations

$$\begin{cases} a + b + c = 0, \\ -4a - b = 2, \\ 16a + b = 10. \end{cases}$$

To solve the system, add the second equation to the third one, then obtain the solution by back substitution:

$$\begin{cases} a + b + c = 0 \\ -4a - b = 2 \\ 16a + b = 10 \end{cases} \Longleftrightarrow \begin{cases} a + b + c = 0 \\ -4a - b = 2 \\ 12a = 12 \end{cases} \Longleftrightarrow \begin{cases} a + b + c = 0 \\ -4a - b = 2 \\ a = 1 \end{cases} \Longleftrightarrow \begin{cases} a + b + c = 0 \\ b = -6 \\ a = 1 \end{cases} \Longleftrightarrow \begin{cases} c = 5 \\ b = -6 \\ a = 1 \end{cases}$$

Thus $f(x) = \cos 2x - 6\cos x + 5$.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}$. Find the inverse matrix A^{-1}.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A\,|\,I) = \left(\begin{array}{cccc|cccc} 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Interchange the first row with the fourth row:

$$\left(\begin{array}{cccc|cccc} 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Subtract 2 times the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 0 & 1 & 0 & -2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Subtract the first row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 0 & 1 & 0 & -2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

At this point, it is convenient to add the fourth row to the second row (rather than divide the second row by 3):

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 3 & 2 & -2 & 0 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right).$$

Add 2 times the second row to the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 16 & -1 & 3 & 2 & 0 & -4 \end{array}\right).$$

Add 16 times the third row to the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 16 & -1 & 3 & 2 & 0 & -4 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 3 & 2 & 16 & -20 \end{array}\right).$$

Multiply the third and the fourth rows by -1:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 3 & 2 & 16 & -20 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right).$$

Add the fourth row to the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & -1 & 1 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & 0 & -2 & -1 & -16 & 18 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right).$$

Subtract the fourth row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 6 & 0 & -2 & -1 & -16 & 18 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 6 & 0 & -2 & -1 & -16 & 18 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right).$$

Subtract 6 times the third row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 6 & 0 & -2 & -1 & -16 & 18 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -3 & -2 & -16 & 20 \end{array}\right).$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -3 & -2 & -16 & 20 \end{pmatrix}.$$

Problem 4 (20 pts.) Evaluate the following determinants:

(i) $\begin{vmatrix} 0 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 1 & 0 & -1 & 1 \\ 1 & 0 & 0 & 1 \end{vmatrix}$.

Let A denote the above matrix. In the solution of Problem 3, the matrix A has been transformed into the identity matrix using elementary row operations. The latter included one row exchange and two row multiplications, each time by -1. It follows that $\det I = -(-1)^2 \det A$. Therefore $\det A = -\det I = -1$.

(ii) $\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix}$.

Expand the determinant by the third column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}.$$

Subtract 2 times the second row from the first row:

$$-2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}.$$

Expand the determinant by the first row:

$$-2\begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2\cdot 9\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}.$$

Finally,

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -18\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18\cdot 2 = -36.$$

Bonus Problem 5 (15 pts.) Find the volume of the tetrahedron with vertices at the points $\mathbf{a} = (1, 0, 0)$, $\mathbf{b} = (0, 1, 0)$, $\mathbf{c} = (0, 0, 1)$, and $\mathbf{d} = (2, 3, 5)$.

The vectors $\mathbf{x} = \mathbf{b} - \mathbf{a} = (-1, 1, 0)$, $\mathbf{y} = \mathbf{c} - \mathbf{a} = (-1, 0, 1)$, and $\mathbf{z} = \mathbf{d} - \mathbf{a} = (1, 3, 5)$ are represented by adjacent edges of the tetrahedron. The volume of a parallelepiped with such adjacent edges equals $|\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})|$. The volume of the tetrahedron is $1/6$ of the volume of the parallelepiped.

We have

$$\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z}) = \begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 3 & 5 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 1 \\ 3 & 5 \end{vmatrix} - 1\begin{vmatrix} -1 & 1 \\ 1 & 5 \end{vmatrix} = (-1)(-3) - 1(-6) = 9.$$

Thus the volume of the tetrahedron is $\frac{1}{6}|\mathbf{x}\cdot(\mathbf{y}\times\mathbf{z})| = \frac{1}{6}\cdot 9 = 1.5$.