MATH 311–504 February 15, 2008

Test 1: Solutions

Problem 1 (25 pts.) Let ℓ_0 be the line in $\mathbb{R}^3$ passing through the point $\mathbf{a} = (1,1,0)$ in the direction $\mathbf{v} = (1,1,1)$. Let Π be the plane in $\mathbb{R}^3$ passing through the line ℓ_0 and the point $\mathbf{b} = (0,1,1)$. Let ℓ be the line in $\mathbb{R}^3$ passing through the points $\mathbf{c} = (1,0,1)$ and $\mathbf{d} = (2,0,2)$.

(i) Find a parametric representation for the line ℓ.

Solution: $s(1,0,1)$.

Since the points $\mathbf{c}$ and $\mathbf{d}$ lie on the line ℓ, the vector $\mathbf{d} - \mathbf{c} = (1,0,1)$ is parallel to ℓ. Hence a parametric representation $t(\mathbf{d} - \mathbf{c}) + \mathbf{c} = t(1,0,1) + (1,0,1)$. Note that the line ℓ passes through the origin (take $t = -1$). Therefore an equivalent representation is $s(1,0,1)$.

(ii) Find a parametric representation for the plane Π.

Solution: $t_1(1,1,1) + t_2(-1,0,1) + (1,1,0)$.

We know that the vector $\mathbf{v}$ is parallel to Π. Besides, the plane contains the points $\mathbf{a}$ and $\mathbf{b}$ so that the vector $\mathbf{b} - \mathbf{a}$ is also parallel to Π. Clearly, the vectors $\mathbf{v} = (1,1,1)$ and $\mathbf{b} - \mathbf{a} = (-1,0,1)$ are not parallel to each other. This leads to a parametric representation $t_1\mathbf{v} + t_2(\mathbf{b} - \mathbf{a}) + \mathbf{a} = t_1(1,1,1) + t_2(-1,0,1) + (1,1,0)$.

(iii) Find an equation for the plane Π.

Solution: $x - 2y + z = -1$.

Since the vectors $\mathbf{v} = (1,1,1)$ and $\mathbf{b} - \mathbf{a} = (-1,0,1)$ are parallel to the plane Π, their cross product $\mathbf{p} = \mathbf{v} \times (\mathbf{b} - \mathbf{a})$ is orthogonal to Π. We have that

$$\mathbf{p} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ -1 & 0 \end{vmatrix} \mathbf{k} = \mathbf{i} - 2\mathbf{j} + \mathbf{k} = (1,-2,1).$$

A point $\mathbf{x} = (x,y,z)$ is in the plane Π if and only if $\mathbf{p} \cdot (\mathbf{x} - \mathbf{a}) = 0$. This is an equation for the plane. In coordinate form, $(x-1) - 2(y-1) + z = 0$ or $x - 2y + z = -1$.

(iv) Find the point where the line ℓ intersects the plane Π.

Solution: $(-1/2, 0, -1/2)$.

Let $\mathbf{x} = (x,y,z)$ be the point of intersection. Then $\mathbf{x} = s(1,0,1)$ for some $s \in \mathbb{R}$ and also $x - 2y + z = -1$. Both conditions are satisfied if and only if $s = -1/2$. Hence $\mathbf{x} = (-1/2, 0, -1/2)$.

(v) Find the angle between the line ℓ and the plane Π.

Solution: $\arcsin \dfrac{1}{\sqrt{3}}$.

Let ϕ denote the angle between the vectors $\mathbf{d} - \mathbf{c} = (1,0,1)$ and $\mathbf{p} = (1,-2,1)$. Then

$$\cos\phi = \frac{(\mathbf{d} - \mathbf{c}) \cdot \mathbf{p}}{|\mathbf{d} - \mathbf{c}|\,|\mathbf{p}|} = \frac{1 \cdot 1 + 0 \cdot (-2) + 1 \cdot 1}{\sqrt{1^2 + 0^2 + 1^2}\,\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{2}{\sqrt{2}\,\sqrt{6}} = \frac{1}{\sqrt{3}}.$$

Note that $0 < \phi < \pi/2$ as $\cos\phi > 0$. Since the vector $\mathbf{d} - \mathbf{c}$ is parallel to the line ℓ while the vector $\mathbf{p}$ is orthogonal to the plane Π, the angle between ℓ and Π is equal to

$$\frac{\pi}{2} - \phi = \frac{\pi}{2} - \arccos\frac{1}{\sqrt{3}} = \arcsin\frac{1}{\sqrt{3}}.$$

(vi) Find the distance from the point $(1, 1, 1)$ to the plane Π.

Solution: $\dfrac{1}{\sqrt{6}}$.

The plane Π can be defined by the equation $x - 2y + z = -1$. Hence the distance from a point (x_0, y_0, z_0) to Π is equal to

$$\frac{|x_0 - 2y_0 + z_0 + 1|}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{|x_0 - 2y_0 + z_0 + 1|}{\sqrt{6}}.$$

In particular, the distance from the point $(1, 1, 1)$ to the plane is equal to $\dfrac{1}{\sqrt{6}}$.

Problem 2 (15 pts.) Find a quadratic polynomial $p(x)$ such that $p(-1) = 1$, $p(2) = -2$, and $p(3) = 1$.

Solution: $p(x) = x^2 - 2x - 2$.

Let $p(x) = ax^2 + bx + c$, where a, b, c are unknown coefficients. Then $p(-1) = a - b + c$, $p(2) = 4a + 2b + c$, and $p(3) = 9a + 3b + c$. The coefficients a, b, and c are to be chosen so that

$$\begin{cases} a - b + c = 1, \\ 4a + 2b + c = -2, \\ 9a + 3b + c = 1. \end{cases}$$

We solve this system of linear equations using elementary operations:

$$\begin{cases} a - b + c = 1 \\ 4a + 2b + c = -2 \\ 9a + 3b + c = 1 \end{cases} \iff \begin{cases} a - b + c = 1 \\ 3a + 3b = -3 \\ 9a + 3b + c = 1 \end{cases} \iff \begin{cases} a - b + c = 1 \\ 3a + 3b = -3 \\ 8a + 4b = 0 \end{cases} \iff \begin{cases} a - b + c = 1 \\ a + b = -1 \\ 2a + b = 0 \end{cases}$$

$$\iff \begin{cases} a - b + c = 1 \\ a + b = -1 \\ a = 1 \end{cases} \iff \begin{cases} a - b + c = 1 \\ b = -2 \\ a = 1 \end{cases} \iff \begin{cases} c = -2 \\ b = -2 \\ a = 1 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - 2x - 2$.

Problem 3 (20 pts.) Let $A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}$. Find the inverse matrix A^{-1}.

Solution: $A^{-1} = \begin{pmatrix} -2 & 3 & 1 & 2 \\ 2 & -2 & -1 & -2 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

First we merge the matrix A with the identity matrix into one 4×8 matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract the first row from the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Interchange the second row with the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Add the second row to the fourth row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Multiply the third row by -1:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Subtract the fourth row from the third row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Subtract the fourth row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Add 2 times the third row to the second row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & -2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Subtract the third row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Subtract the second row from the first row:

$$\left(\begin{array}{cccc|cccc} 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right) \to \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -2 & 3 & 1 & 2 \\ 0 & 1 & 0 & 0 & 2 & -2 & -1 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \end{array}\right).$$

Finally the left part of our 4×8 matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of A. Thus

$$A^{-1} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 2 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 3 & 1 & 2 \\ 2 & -2 & -1 & -2 \\ 1 & -1 & -1 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Problem 4 (20 pts.) Let A be the same matrix as in Problem 3. Evaluate the following determinants:

(i) $\det A$;
(ii) $\det(A - I)$;
(iii) $\det(2A)$.

Solution: $\det A = \det(A - I) = 1$, $\det(2A) = 16$.

In the solution of Problem 3, the matrix A has been transformed into the identity matrix using elementary row operations. The latter included one row exchange and one row multiplications by -1. It follows that $\det I = -(-1)\det A$. Therefore $\det A = \det I = 1$.

Since A is a 4×4 matrix, we have that $\det(2A) = 2^4 \det A = 16 \det A = 16$.

The determinant of the matrix

$$A - I = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 2 & 0 \end{pmatrix}$$

is easily evaluated using column expansions:

$$\det(A - I) = \begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 1 & 1 \\ 1 & -3 & 0 \\ -1 & 2 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & -3 \\ -1 & 2 \end{vmatrix} = -(-1) = 1.$$

Bonus Problem 5 (20 pts.) Let $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (0, 1, 1)$, $\mathbf{v}_3 = (1, 1, 1)$, and $\mathbf{v}_4 = (0, 1, 0)$. Determine which of the following sets of vectors are linearly independent:

(i) $\mathbf{v}_1$, $\mathbf{v}_2$;
(ii) $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$;
(iii) $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{v}_4$.

Solution: The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are also linearly independent. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent.

The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent since they are not scalar multiples of each other.

Consider the 3×3 matrix V whose rows are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. We obtain

$$\det V = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1.$$

Since $\det V \neq 0$, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are linearly independent.

Finally, the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ are linearly dependent as there are more vectors in this set than coordinates. Besides, it is easy to observe a nontrivial linear relation between these vectors: $\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3 - \mathbf{v}_4 = \mathbf{0}$.