MATH 311–504 March 28, 2008

Test 2: Solutions

Problem 1 (20 pts.) Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set S_1 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x - y + 2z = 0$.
(ii) The set S_2 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + 2y + 3z = 6$.
(iii) The set S_3 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y = z^2$.
(iv) The set S_4 of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 + z^2 = 0$.

Solution: S_1 and S_4.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, a subspace must not be empty.

The set S_1 is a plane passing through the origin. It is closed under addition and scalar multiplication.

S_2 is a plane that does not pass through the origin. It is not closed under scalar multiplication as the following example shows: $(1, 1, 1) \in S_2$ but $0(1, 1, 1) = (0, 0, 0) \notin S_2$.

S_3 is a parabolic cylinder. It is not closed under scalar multiplication as the following example shows: $(0, 1, 1) \in S_3$ but $2(0, 1, 1) = (0, 2, 2) \notin S_3$.

The condition $x^2 + y^2 + z^2 = 0$ is equivalent to $x = y = z = 0$. Hence the set S_4 contains only the zero vector. Clearly, it is a subspace.

Thus S_1 and S_4 are subspaces of $\mathbb{R}^3$ while S_2 and S_3 are not.

Problem 2 (20 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the space of 2-by-2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \to \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Solution: $\begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$.

Let M_L denote the desired matrix. By definition, M_L is a 4-by-4 matrix whose columns are coordinates of the matrices $L(E_1), L(E_2), L(E_3), L(E_4)$ with respect to the basis E_1, E_2, E_3, E_4. We have that

$$L\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} y & x \\ w & z \end{pmatrix} = \begin{pmatrix} y+2w & x+2z \\ w & z \end{pmatrix}.$$

In particular,

$$L(E_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0E_1 + 1E_2 + 0E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1E_1 + 0E_2 + 0E_3 + 0E_4,$$

$$L(E_3) = \begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = 0E_1 + 2E_2 + 0E_3 + 1E_4,$$

$$L(E_4) = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} = 2E_1 + 0E_2 + 1E_3 + 0E_4.$$

It follows that

$$M_L = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Problem 3 (30 pts.) Consider a linear operator $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(\mathbf{x}) = A\mathbf{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{pmatrix}.$$

(i) Find a basis for the image of f.

Solution: $(1, -1, 2)$, $(1, 0, 1)$.

The image of the linear operator f is the subspace of $\mathbb{R}^3$ spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (1, -3, 4)$. The third column is a linear combination of the first two, $\mathbf{v}_3 = 3\mathbf{v}_1 - 2\mathbf{v}_2$ (this relation can be found using the method of undetermined coefficients; one has to solve a system of linear equations). Therefore the span of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ is the same as the span of $\mathbf{v}_1$ and $\mathbf{v}_2$. The vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because they are not parallel. Thus $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

Alternative solution: The image of f is spanned by columns of the matrix A, that is, by vectors $\mathbf{v}_1 = (1, -1, 2)$, $\mathbf{v}_2 = (1, 0, 1)$, and $\mathbf{v}_3 = (1, -3, 4)$. To check linear independence of these vectors, we evaluate the determinant of A (using expansion by the second column):

$$\det A = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{vmatrix} = -1 \begin{vmatrix} -1 & -3 \\ 2 & 4 \end{vmatrix} - 1 \begin{vmatrix} 1 & 1 \\ -1 & -3 \end{vmatrix} = -1 \cdot 2 - 1 \cdot (-2) = 0.$$

Since $\det A = 0$, the columns of the matrix A are linearly dependent. Then the image of f is at most two-dimensional. On the other hand, the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly independent because they are not parallel. Hence they span a two-dimensional subspace of $\mathbb{R}^3$. It follows that this subspace coincides with the image of f. Therefore $\mathbf{v}_1$, $\mathbf{v}_2$ is a basis for the image of f.

(ii) Find a basis for the null-space of f.

Solution: $(-3, 2, 1)$.

The null-space of f is the set of solutions of the vector equation $A\mathbf{x} = \mathbf{0}$. To solve the equation, we shall convert the matrix A to reduced row echelon form. Since the right-hand side of the equation is the zero vector, elementary row operations do not change the solution set.

First we add the first row of the matrix A to the second row and subtract it twice from the third row:

$$\begin{pmatrix} 1 & 1 & 1 \\ -1 & 0 & -3 \\ 2 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 2 & 1 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix}.$$

Then we add the second row to the third row:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & -1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Finally, we subtract the second row from the first row:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}.$$

It follows that the vector equation $A\mathbf{x} = \mathbf{0}$ is equivalent to the system $x + 3z = y - 2z = 0$, where $\mathbf{x} = (x, y, z)$. The general solution of the system is $x = -3t$, $y = 2t$, $z = t$ for an arbitrary $t \in \mathbb{R}$. That is, $\mathbf{x} = (-3t, 2t, t) = t(-3, 2, 1)$, where $t \in \mathbb{R}$. Thus the null-space of the linear operator f is the line $t(-3, 2, 1)$. The vector $(-3, 2, 1)$ is a basis for this line.

Problem 4 (30 pts.) Let $B = \begin{pmatrix} -1 & 1 \\ 5 & 3 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.

Solution: -2 and 4.

The eigenvalues of B are roots of the characteristic equation $\det(B - \lambda I) = 0$. We obtain that

$$\det(B - \lambda I) = \begin{vmatrix} -1-\lambda & 1 \\ 5 & 3-\lambda \end{vmatrix} = (-1-\lambda)(3-\lambda) - 5 = \lambda^2 - 2\lambda - 8 = (\lambda - 4)(\lambda + 2).$$

Hence the matrix B has two eigenvalues: -2 and 4.

(ii) For each eigenvalue of B, find an associated eigenvector.

Solution: $\mathbf{v}_1 = (-1, 1)$ and $\mathbf{v}_2 = (1, 5)$ are eigenvectors of B associated with the eigenvalues -2 and 4, respectively.

An eigenvector $\mathbf{v} = (x, y)$ of B associated with an eigenvalue λ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{v} = \mathbf{0}$.

First consider the case $\lambda = -2$. We obtain

$$(B + 2I)\mathbf{v} = \mathbf{0} \quad \Longleftrightarrow \quad \begin{pmatrix} 1 & 1 \\ 5 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longleftrightarrow \quad x + y = 0.$$

The general solution is $x = -t$, $y = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_1 = (-1, 1)$ is an eigenvector of B associated with the eigenvalue -2.

Now consider the case $\lambda = 4$. We obtain

$$(B-4I)\mathbf{v}=\mathbf{0} \iff \begin{pmatrix} -5 & 1 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff -5x+y=0.$$

The general solution is $x=t$, $y=5t$, where $t\in\mathbb{R}$. In particular, $\mathbf{v}_2=(1,5)$ is an eigenvector of B associated with the eigenvalue 4.

(iii) Is there a basis for $\mathbb{R}^2$ consisting of eigenvectors of B? Explain.

Solution: Yes.

By the above the vectors $\mathbf{v}_1=(-1,1)$ and $\mathbf{v}_2=(1,5)$ are eigenvectors of the matrix B. These vectors are linearly independent since they are not parallel. It follows that $\mathbf{v}_1,\mathbf{v}_2$ is a basis for $\mathbb{R}^2$.

Alternatively, the existence of a basis for $\mathbb{R}^2$ consisting of eigenvectors of B already follows from the fact that the matrix B has two distinct eigenvalues.

(iv) Find all eigenvalues of the matrix B^2.

Solution: 4 and 16.

Suppose that $B\mathbf{v}=\lambda\mathbf{v}$ for some $\mathbf{v}\in\mathbb{R}^2$ and $\lambda\in\mathbb{R}$. Then

$$B^2\mathbf{v}=B(B\mathbf{v})=B(\lambda\mathbf{v})=\lambda(B\mathbf{v})=\lambda^2\mathbf{v}.$$

It follows that the vectors $\mathbf{v}_1=(-1,1)$ and $\mathbf{v}_2=(1,5)$ are eigenvectors of the matrix B^2 associated with eigenvalues $(-2)^2=4$ and $4^2=16$, respectively. Since a 2-by-2 matrix can have at most 2 eigenvalues, 4 and 16 are the only eigenvalues of B^2.

Bonus Problem 5 (20 pts.) Solve the following system of differential equations (find all solutions):

$$\begin{cases} \dfrac{dx}{dt} = -x+y, \\ \dfrac{dy}{dt} = 5x+3y. \end{cases}$$

Solution: $x(t)=-c_1e^{-2t}+c_2e^{4t}$, $y(t)=c_1e^{-2t}+5c_2e^{4t}$, where c_1,c_2 are arbitrary constants.

Introducing a vector function $\mathbf{v}(t)=(x(t),y(t))$, we can rewrite the system in the following way:

$$\frac{d\mathbf{v}}{dt}=B\mathbf{v}, \quad \text{where} \quad B=\begin{pmatrix} -1 & 1 \\ 5 & 3 \end{pmatrix}.$$

As shown in the solution of Problem 4, there is a basis for $\mathbb{R}^2$ consisting of eigenvectors of the matrix B. Namely, $\mathbf{v}_1=(-1,1)$ and $\mathbf{v}_2=(1,5)$ are eigenvectors of B associated with the eigenvalues -2 and 4, respectively. These vectors form a basis for $\mathbb{R}^2$. It follows that

$$\mathbf{v}(t)=r_1(t)\mathbf{v}_1+r_2(t)\mathbf{v}_2,$$

where r_1,r_2 are well-defined scalar functions (coordinates of $\mathbf{v}$ with respect to the basis $\mathbf{v}_1,\mathbf{v}_2$). Then

$$\frac{d\mathbf{v}}{dt}=\frac{dr_1}{dt}\mathbf{v}_1+\frac{dr_2}{dt}\mathbf{v}_2, \qquad B\mathbf{v}=r_1B\mathbf{v}_1+r_2B\mathbf{v}_2=-2r_1\mathbf{v}_1+4r_2\mathbf{v}_2.$$

As a consequence,

$$\frac{d\mathbf{v}}{dt} = B\mathbf{v} \quad \Longleftrightarrow \quad \begin{cases} \frac{dr_1}{dt} = -2r_1, \\ \frac{dr_2}{dt} = 4r_2. \end{cases}$$

The general solution of the differential equation $r_1' = -2r_1$ is $r_1(t) = c_1 e^{-2t}$, where c_1 is an arbitrary constant. The general solution of the equation $r_2' = 4r_2$ is $r_2(t) = c_2 e^{4t}$, where c_2 is another arbitrary constant. Therefore the general solution of the equation $\mathbf{v}' = B\mathbf{v}$ is

$$\mathbf{v}(t) = c_1 e^{-2t}\mathbf{v}_1 + c_2 e^{4t}\mathbf{v}_2 = c_1 e^{-2t}\begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 e^{4t}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} -c_1 e^{-2t} + c_2 e^{4t} \\ c_1 e^{-2t} + 5c_2 e^{4t} \end{pmatrix},$$

where $c_1, c_2 \in \mathbb{R}$. Equivalently,

$$\begin{cases} x(t) = -c_1 e^{-2t} + c_2 e^{4t}, \\ y(t) = c_1 e^{-2t} + 5c_2 e^{4t}. \end{cases}$$