MATH 311–505/506 Fall 2009

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Find a quadratic polynomial $p(x) = ax^2 + bx + c$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

Problem 2 (20 pts.) Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
(iii) Find the matrix of the operator L relative to the standard basis.

Problem 3 (20 pts.) Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B?
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B?

Problem 4 (20 pts.) Find a quadratic polynomial q that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$. This means that q should minimize the distance

$$\operatorname{dist}(f, q) = \left(\int_{-1}^{1} |f(x) - q(x)|^2 \, dx \right)^{1/2}$$

over all polynomials of degree at most 2.

Problem 5 (25 pts.) It is known that

$$\int x^2 \sin(ax) \, dx = \left(-\frac{x^2}{a} + \frac{2}{a^3} \right) \cos(ax) + \frac{2x}{a^2} \sin(ax) + C, \quad a \neq 0.$$

(i) Find the Fourier sine series of the function $f(x) = x^2$ on the interval $[0, \pi]$.
(ii) Over the interval $[-3.5\pi, 3.5\pi]$, sketch the function to which the series converges.
(iii) Describe how the answer to (ii) would change if we studied the Fourier cosine series instead.

Bonus Problem 6 (15 pts.) Solve the initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad t > 0),$$

$$u(x, 0) = 1 + 2\cos(2x) - \cos(3x) \qquad (0 < x < \pi),$$

$$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(\pi, t) = 0 \qquad (t > 0).$$