MATH 311–505/506 Fall 2009

# Sample problems for the final exam

**Any problem may be altered or replaced by a different one!**

**Problem 1 (15 pts.)** Find a quadratic polynomial $p(x) = ax^2 + bx + c$ such that $p(-1) = p(3) = 6$ and $p'(2) = p(1)$.

We have that $p(x) = ax^2+bx+c$. Then $p(-1) = a-b+c$, $p(1) = a+b+c$, and $p(3) = 9a+3b+c$. Also, $p'(x) = 2ax + b$ and $p'(2) = 4a + b$. The coefficients $a$, $b$, and $c$ have to be chosen so that

$$\begin{cases} a-b+c=6, \\ 9a+3b+c=6, \\ 4a+b=a+b+c. \end{cases}$$

This is a system of linear equations in variables $a, b, c$. To solve the system, let us convert the third equation to the standard form and add it to the first and the second equations:

$$\begin{cases} a-b+c=6 \\ 9a+3b+c=6 \\ 3a-c=0 \end{cases} \iff \begin{cases} 4a-b=6 \\ 9a+3b+c=6 \\ 3a-c=0 \end{cases} \iff \begin{cases} 4a-b=6 \\ 12a+3b=6 \\ 3a-c=0 \end{cases}$$

Now divide the second equation by 3, add it to the first equation, and find the solution by back substitution:

$$\begin{cases} 4a-b=6 \\ 4a+b=2 \\ 3a-c=0 \end{cases} \iff \begin{cases} 8a=8 \\ 4a+b=2 \\ 3a-c=0 \end{cases} \iff \begin{cases} a=1 \\ 4a+b=2 \\ 3a-c=0 \end{cases} \iff \begin{cases} a=1 \\ b=-2 \\ 3a-c=0 \end{cases} \iff \begin{cases} a=1 \\ b=-2 \\ c=3 \end{cases}$$

Thus the desired polynomial is $p(x) = x^2 - 2x + 3$.

**Problem 2 (20 pts.)** Let $\mathbf{v}_1 = (1,1,1)$, $\mathbf{v}_2 = (1,1,0)$, and $\mathbf{v}_3 = (1,0,1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.

Let $U$ be a $3\times 3$ matrix such that its columns are vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$:

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the determinant of $U$, we subtract the second row from the first one and then expand by the first row:

$$\det U = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1.$$

Since $\det U \neq 0$, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent. It follows that they form a basis for $\mathbb{R}^3$.

(ii) Find the matrix of the operator $L$ relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

Let $A$ denote the matrix of $L$ relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. By definition, the columns of $A$ are coordinates of vectors $L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)$ with respect to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Since $L(\mathbf{v}_1) = \mathbf{v}_2 = 0\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3$, $L(\mathbf{v}_2) = \mathbf{v}_3 = 0\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 0\mathbf{v}_3$, we obtain

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

(iii) Find the matrix of the operator $L$ relative to the standard basis.

Let $S$ denote the matrix of $L$ relative to the standard basis for $\mathbb{R}^3$. We have $S = UAU^{-1}$, where $A$ is the matrix of $L$ relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ (already found) and $U$ is the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to the standard basis (the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are consecutive columns of $U$):

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad U = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

To find the inverse $U^{-1}$, we merge the matrix $U$ with the identity matrix $I$ into one $3 \times 6$ matrix and apply row reduction to convert the left half $U$ of this matrix into $I$. Simultaneously, the right half $I$ will be converted into $U^{-1}$:

$$(U|I) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 & -1 & 0 \end{array}\right) = (I|U^{-1}).$$

Thus

$$S = UAU^{-1} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

*Alternative solution:* Let $S$ denote the matrix of $L$ relative to the standard basis $\mathbf{e}_1 = (1,0,0), \mathbf{e}_2 = (0,1,0), \mathbf{e}_3 = (0,0,1)$. By definition, the columns of $S$ are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), L(\mathbf{e}_3)$. It is easy to observe that $\mathbf{e}_2 = \mathbf{v}_1 - \mathbf{v}_3$, $\mathbf{e}_3 = \mathbf{v}_1 - \mathbf{v}_2$, and $\mathbf{e}_1 = \mathbf{v}_2 - \mathbf{e}_2 = -\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$. Therefore

$$L(\mathbf{e}_1) = L(-\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) = -L(\mathbf{v}_1) + L(\mathbf{v}_2) + L(\mathbf{v}_3) = -\mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_1 = (1,0,2),$$
$$L(\mathbf{e}_2) = L(\mathbf{v}_1 - \mathbf{v}_3) = L(\mathbf{v}_1) - L(\mathbf{v}_3) = \mathbf{v}_2 - \mathbf{v}_1 = (0,0,-1),$$
$$L(\mathbf{e}_3) = L(\mathbf{v}_1 - \mathbf{v}_2) = L(\mathbf{v}_1) - L(\mathbf{v}_2) = \mathbf{v}_2 - \mathbf{v}_3 = (0,1,-1).$$

Thus

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & -1 \end{pmatrix}.$$

**Problem 3 (20 pts.)** Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix $B$.

The eigenvalues of $B$ are roots of the characteristic equation $\det(B - \lambda I) = 0$. One obtains that

$$\det(B - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^3 - 3(1-\lambda) + 2$$

$$= (1 - 3\lambda + 3\lambda^2 - \lambda^3) - 3(1-\lambda) + 2 = 3\lambda^2 - \lambda^3 = \lambda^2(3-\lambda).$$

Hence the matrix $B$ has two eigenvalues: 0 and 3.

(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of $B$?

An eigenvector $\mathbf{x} = (x, y, z)$ of $B$ associated with an eigenvalue $\lambda$ is a nonzero solution of the vector equation $(B - \lambda I)\mathbf{x} = \mathbf{0}$. First consider the case $\lambda = 0$. We obtain that

$$B\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff x + y + z = 0.$$

The general solution is $x = -t - s$, $y = t$, $z = s$, where $t, s \in \mathbb{R}$. Equivalently, $\mathbf{x} = t(-1, 1, 0) + s(-1, 0, 1)$. Hence the eigenspace of $B$ associated with the eigenvalue 0 is two-dimensional. It is spanned by eigenvectors $\mathbf{v}_1 = (-1, 1, 0)$ and $\mathbf{v}_2 = (-1, 0, 1)$.

Now consider the case $\lambda = 3$. We obtain that

$$(B - 3I)\mathbf{x} = \mathbf{0} \iff \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - y = 0, \\ y - z = 0. \end{cases}$$

The general solution is $x = y = z = t$, where $t \in \mathbb{R}$. In particular, $\mathbf{v}_3 = (1, 1, 1)$ is an eigenvector of $B$ associated with the eigenvalue 3.

The vectors $\mathbf{v}_1 = (-1, 1, 0)$, $\mathbf{v}_2 = (-1, 0, 1)$, and $\mathbf{v}_3 = (1, 1, 1)$ are eigenvectors of the matrix $B$. They are linearly independent since the matrix whose rows are these vectors is invertible:

$$\begin{vmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 3 \neq 0.$$

It follows that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is a basis for $\mathbb{R}^3$.

(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of $B$?

It is easy to check that the vector $\mathbf{v}_3$ is orthogonal to $\mathbf{v}_1$ and $\mathbf{v}_2$. To transform the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into an orthogonal one, we only need to orthogonalize the pair $\mathbf{v}_1, \mathbf{v}_2$. Namely, we replace the vector $\mathbf{v}_2$ by

$$\mathbf{u} = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = (-1,0,1) - \frac{1}{2}(-1,1,0) = (-1/2,-1/2,1).$$

Now $\mathbf{v}_1, \mathbf{u}, \mathbf{v}_3$ is an orthogonal basis for $\mathbb{R}^3$. Since $\mathbf{u}$ is a linear combination of the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, it is also an eigenvector of $B$ associated with the eigenvalue 0.

Finally, vectors $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{|\mathbf{v}_1|}$, $\mathbf{w}_2 = \dfrac{\mathbf{u}}{|\mathbf{u}|}$, and $\mathbf{w}_3 = \dfrac{\mathbf{v}_3}{|\mathbf{v}_3|}$ form an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of $B$. We get that $|\mathbf{v}_1| = \sqrt{2}$, $|\mathbf{u}| = \sqrt{3/2}$, and $|\mathbf{v}_3| = \sqrt{3}$. Thus

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(-1,1,0), \quad \mathbf{w}_2 = \frac{1}{\sqrt{6}}(-1,-1,2), \quad \mathbf{w}_3 = \frac{1}{\sqrt{3}}(1,1,1).$$

**Problem 4 (20 pts.)** Find a quadratic polynomial $q$ that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1,1]$. This means that $q$ should minimize the distance

$$\operatorname{dist}(f,q) = \left(\int_{-1}^{1} |f(x) - q(x)|^2\,dx\right)^{1/2}$$

over all polynomials of degree at most 2.

The above distance on $C[-1,1]$ is induced by the norm

$$\|g\| = \left(\int_{-1}^{1} |g(x)|^2\,dx\right)^{1/2},$$

which, in turn, is induced by the inner product

$$\langle g,h\rangle = \int_{-1}^{1} g(x)h(x)\,dx.$$

It follows that the best least squares fit $q$ is the orthogonal projection (relative to this inner product) of the function $f$ onto the subspace $\mathcal{P}_3$ of polynomials of degree less than 3. Suppose that $p_0, p_1, p_2$ is an orthogonal basis for $\mathcal{P}_3$. Then

$$q(x) = \frac{\langle f,p_0\rangle}{\langle p_0,p_0\rangle}p_0(x) + \frac{\langle f,p_1\rangle}{\langle p_1,p_1\rangle}p_1(x) + \frac{\langle f,p_2\rangle}{\langle p_2,p_2\rangle}p_2(x).$$

To get an orthogonal basis for the subspace $\mathcal{P}_3$, we apply the Gram-Schmidt orthogonalization process to the basis $1$, $x$, $x^2$:

$$\begin{aligned}
p_0(x) &= 1,\\
p_1(x) &= x - \frac{\langle x,p_0\rangle}{\langle p_0,p_0\rangle}p_0(x),\\
p_2(x) &= x^2 - \frac{\langle x^2,p_0\rangle}{\langle p_0,p_0\rangle}p_0(x) - \frac{\langle x^2,p_1\rangle}{\langle p_1,p_1\rangle}p_1(x).
\end{aligned}$$

Note that

$$\langle x, p_0 \rangle = \int_{-1}^{1} x\,dx = 0.$$

Hence $p_1(x) = x$. Furthermore,

$$\langle p_0, p_0 \rangle = \int_{-1}^{1} dx = 2,$$

$$\langle x^2, p_0 \rangle = \langle p_1, p_1 \rangle = \int_{-1}^{1} x^2\,dx = \frac{2}{3},$$

$$\langle x^2, p_1 \rangle = \int_{-1}^{1} x^3\,dx = 0.$$

It follows that $p_2(x) = x^2 - 1/3$.

Now we can start computing the orthogonal projection of $f$ onto $\mathcal{P}_3$:

$$\langle f, p_0 \rangle = \int_{-1}^{1} |x|\,dx = 2\int_{0}^{1} x\,dx = 1,$$

$$\langle f, p_1 \rangle = \int_{-1}^{1} |x|x\,dx = 0,$$

$$\langle f, p_2 \rangle = \int_{-1}^{1} |x|\Big(x^2 - \frac{1}{3}\Big)\,dx = 2\int_{0}^{1} \Big(x^3 - \frac{1}{3}x\Big)\,dx = \frac{1}{6},$$

$$\langle p_2, p_2 \rangle = \int_{-1}^{1} \Big(x^2 - \frac{1}{3}\Big)^2\,dx = 2\int_{0}^{1} \Big(x^4 - \frac{2}{3}x^2 + \frac{1}{9}\Big)\,dx = \frac{8}{45}.$$

Thus

$$q(x) = \frac{1}{2}p_0(x) + \frac{1/6}{8/45}p_2(x) = \frac{1}{2} + \frac{15}{16}\Big(x^2 - \frac{1}{3}\Big) = \frac{15}{16}x^2 + \frac{3}{16}.$$

**Problem 5 (25 pts.)** It is known that

$$\int x^2 \sin(ax)\,dx = \left(-\frac{x^2}{a} + \frac{2}{a^3}\right)\cos(ax) + \frac{2x}{a^2}\sin(ax) + C, \quad a \neq 0.$$

(i) Find the Fourier sine series of the function $f(x) = x^2$ on the interval $[0, \pi]$.

The required series is $\sum_{n=1}^{\infty} B_n \sin(nx)$, where

$$B_n = \frac{2}{\pi}\int_{0}^{\pi} x^2 \sin(nx)\,dx.$$

Using the given table integral, we obtain

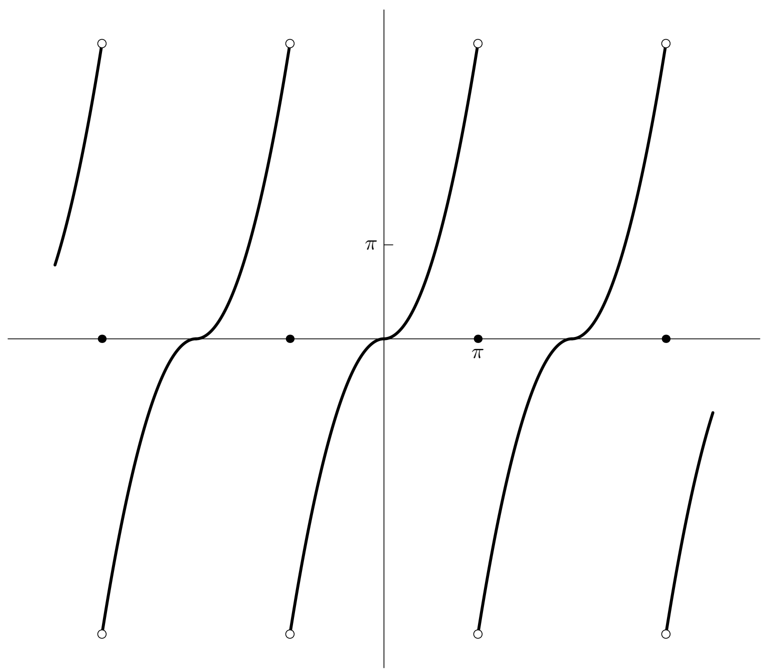
$$B_n = \frac{2}{\pi}\left(-\frac{x^2}{n} + \frac{2}{n^3}\right)\cos(nx)\,\Big|_0^\pi + \frac{2}{\pi}\cdot\frac{2x}{n^2}\sin(nx)\,\Big|_0^\pi$$

$$= \frac{2}{\pi}\left(-\frac{x^2}{n} + \frac{2}{n^3}\right)\cos(nx)\Big|_0^{\pi} = -\frac{2\pi}{n}\cos(n\pi) + \frac{4}{n^3\pi}\big(\cos(n\pi) - 1\big).$$

If $n$ is even, then $\cos(n\pi) = 1$ and $B_n = -2\pi n^{-1}$. If $n$ is odd, then $\cos(n\pi) = -1$ and $B_n = 2\pi n^{-1} - 8\pi^{-1} n^{-3}$.

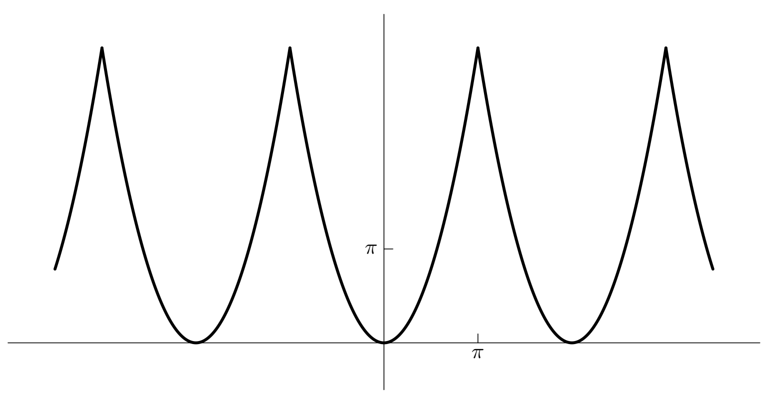
(ii) Over the interval $[-3.5\pi, 3.5\pi]$, sketch the function to which the series converges.

The series converges to an odd $2\pi$-periodic function that coincides with $f$ on the interval $(0, \pi)$. The sum has jump discontinuities at points $\pi + 2k\pi$, $k \in \mathbb{Z}$. The value of the sum at the points of discontinuity is zero.




(iii) Describe how the answer to (ii) would change if we studied the Fourier cosine series instead.

The Fourier cosine series of the function $f(x) = x^2$ on the interval $[0, \pi]$ converges to an even $2\pi$-periodic function that coincides with $f$ on the interval $(0, \pi)$. The sum is continuous and piecewise smooth.



**Bonus Problem 6 (15 pts.)** Solve the initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad (0 < x < \pi, \quad t > 0),$$

$$u(x,0) = 1 + 2\cos(2x) - \cos(3x) \qquad (0 < x < \pi),$$

$$\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(\pi,t) = 0 \qquad (t > 0).$$

We search for the solution of the initial-boundary value problem as a superposition of solutions $u(x,t) = \phi(x)g(t)$ with separated variables of the heat equation that satisfy the boundary conditions. Substituting $u(x,t) = \phi(x)g(t)$ into the heat equation, we obtain

$$\phi(x)g'(t) = \phi''(x)g(t),$$

$$\frac{g'(t)}{g(t)} = \frac{\phi''(x)}{\phi(x)}.$$

Since the left-hand side does not depend on $x$ while the right-hand side does not depend on $t$, it follows that

$$\frac{g'(t)}{g(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda,$$

where $\lambda$ is a constant. Then

$$g' = -\lambda g, \qquad \phi'' = -\lambda\phi.$$

Conversely, if functions $g$ and $\phi$ are solutions of the above ODEs for the same value of $\lambda$, then $u(x,t) = \phi(x)g(t)$ is a solution of the heat equation.

Substituting $u(x,t) = \phi(x)g(t)$ into the boundary conditions, we get

$$\phi'(0)g(t) = \phi'(\pi)g(t) = 0.$$

It is no loss to assume that $g$ is not identically zero. Then the boundary conditions are satisfied if and only if $\phi'(0) = \phi'(\pi) = 0$.

The eigenvalue problem

$$\phi'' = -\lambda\phi, \quad \phi'(0) = \phi'(\pi) = 0$$

has eigenvalues $\lambda_n = n^2$, $n = 0, 1, 2, \dots$. The associated eigenfunctions are $\phi_n(x) = \cos(nx)$. Further, the general solution of the equation $g' = -\lambda g$ is $g(t) = ce^{-\lambda t}$, where $c$ is an arbitrary constant. Thus we obtain the following solutions of the heat equation that satisfy the boundary conditions:

$$u_n(x,t) = e^{-\lambda_n t}\phi_n(x) = e^{-n^2 t}\cos(nx), \quad n = 0, 1, 2, \dots$$

A superposition of these solutions is a series

$$u(x,t) = \sum\nolimits_{n=0}^{\infty} c_n e^{-\lambda_n t}\phi_n(x) = \sum\nolimits_{n=0}^{\infty} c_n e^{-n^2 t}\cos(nx),$$

where $c_1, c_2, \dots$ are constants. Substituting the series into the initial condition, we get

$$1 + 2\cos(2x) - \cos(3x) = \sum\nolimits_{n=0}^{\infty} c_n \cos(nx).$$

It follows that $c_0 = 1$, $c_2 = 2$, $c_3 = -1$ while the other coefficients are zeros. The solution of the initial-boundary value problem is

$$u(x,t) = 1 + 2e^{-4t}\cos(2x) - e^{-9t}\cos(3x).$$