MATH 311

Topics in Applied Mathematics

Lecture 4:
Matrix multiplication.
Diagonal matrices.
Inverse matrix.

Matrices

Definition. An **m-by-n matrix** is a rectangular array of numbers that has m rows and n columns:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

Notation: $A = (a_{ij})_{1 \le i \le n,\, 1 \le j \le m}$ or simply $A = (a_{ij})$ if the dimensions are known.

Matrix algebra: linear operations

Addition: two matrices of the same dimensions can be added by adding their corresponding entries.

Scalar multiplication: to multiply a matrix A by a scalar r, one multiplies each entry of A by r.

Zero matrix O: all entries are zeros.

Negative: $-A$ is defined as $(-1)A$.

Subtraction: $A - B$ is defined as $A + (-B)$.

As far as the linear operations are concerned, the $m \times n$ matrices can be regarded as mn-dimensional vectors.

Properties of linear operations

$$(A + B) + C = A + (B + C)$$

$$A + B = B + A$$

$$A + O = O + A = A$$

$$A + (-A) = (-A) + A = O$$

$$r(sA) = (rs)A$$

$$r(A + B) = rA + rB$$

$$(r + s)A = rA + sA$$

$$1A = A$$

$$0A = O$$

Matrix algebra: matrix multiplication

The product of matrices A and B is defined if the number of columns in A matches the number of rows in B.

Definition. Let $A = (a_{ik})$ be an $m \times n$ matrix and $B = (b_{kj})$ be an $n \times p$ matrix. The **product** AB is defined to be the $m \times p$ matrix $C = (c_{ij})$ such that $\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$ for all indices i, j.

That is, matrices are multiplied **row by column**.

$$A = \left(\begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ \hline a_{21} & a_{22} & \dots & a_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline a_{m1} & a_{m2} & \dots & a_{mn} \end{array}\right) = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix}$$

$$B = \left(\begin{array}{c|c|c|c} b_{11} & b_{12} & \dots & b_{1p} \\ b_{21} & b_{22} & \dots & b_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \dots & b_{np} \end{array}\right) = (\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_p)$$

$$\implies AB = \begin{pmatrix} \mathbf{v}_1 \cdot \mathbf{w}_1 & \mathbf{v}_1 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_1 \cdot \mathbf{w}_p \\ \mathbf{v}_2 \cdot \mathbf{w}_1 & \mathbf{v}_2 \cdot \mathbf{w}_2 & \dots & \mathbf{v}_2 \cdot \mathbf{w}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{v}_m \cdot \mathbf{w}_1 & \mathbf{v}_m \cdot \mathbf{w}_2 & \dots & \mathbf{v}_m \cdot \mathbf{w}_p \end{pmatrix}$$

Examples.

$$(x_1, x_2, \ldots, x_n) \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} = \left(\sum_{k=1}^{n} x_k y_k\right),$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} (x_1, x_2, \ldots, x_n) = \begin{pmatrix} y_1x_1 & y_1x_2 & \ldots & y_1x_n \\ y_2x_1 & y_2x_2 & \ldots & y_2x_n \\ \vdots & \vdots & \ddots & \vdots \\ y_nx_1 & y_nx_2 & \ldots & y_nx_n \end{pmatrix}.$$

Example.

$$\begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 1 & 3 & 0 \\ -3 & 17 & 16 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 3 & 1 & 1 \\ -2 & 5 & 6 & 0 \\ 1 & 7 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix} \text{ is not defined}$$

System of linear equations:

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

Matrix representation of the system:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\left\{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right. \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Properties of matrix multiplication:

$(AB)C = A(BC)$ (associative law)

$(A + B)C = AC + BC$ (distributive law #1)

$C(A + B) = CA + CB$ (distributive law #2)

$(rA)B = A(rB) = r(AB)$

Any of the above identities holds provided that matrix sums and products are well defined.

If A and B are $n \times n$ matrices, then both AB and BA are well defined $n \times n$ matrices.

However, in general, $AB \neq BA$.

Example. Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Then $AB = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $BA = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}$.

If AB does equal BA, we say that the matrices A and B **commute**.

Diagonal matrices

If $A = (a_{ij})$ is a square matrix, then the entries a_{ii} are called **diagonal entries**. A square matrix is called **diagonal** if all non-diagonal entries are zeros.

Example. $\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, denoted $\mathrm{diag}(7, 1, 2)$.

Let $A = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, $B = \mathrm{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \mathrm{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,

$$rA = \mathrm{diag}(rs_1, rs_2, \ldots, rs_n).$$

Example.

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -7 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

Theorem Let $A = \mathrm{diag}(s_1, s_2, \ldots, s_n)$, $B = \mathrm{diag}(t_1, t_2, \ldots, t_n)$.

Then $A + B = \mathrm{diag}(s_1 + t_1, s_2 + t_2, \ldots, s_n + t_n)$,

$$rA = \mathrm{diag}(rs_1, rs_2, \ldots, rs_n).$$

$$AB = \mathrm{diag}(s_1 t_1, s_2 t_2, \ldots, s_n t_n).$$

In particular, diagonal matrices always commute.

Example.

$$\begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} 7a_{11} & 7a_{12} & 7a_{13} \\ a_{21} & a_{22} & a_{23} \\ 2a_{31} & 2a_{32} & 2a_{33} \end{pmatrix}$$

Theorem Let $D = \mathrm{diag}(d_1, d_2, \dots, d_m)$ and A be an $m \times n$ matrix. Then the matrix DA is obtained from A by multiplying the ith row by d_i for $i = 1, 2, \dots, m$:

$$A = \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{pmatrix} \implies DA = \begin{pmatrix} d_1\mathbf{v}_1 \\ d_2\mathbf{v}_2 \\ \vdots \\ d_m\mathbf{v}_m \end{pmatrix}$$

Example.

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 7a_{11} & a_{12} & 2a_{13} \\ 7a_{21} & a_{22} & 2a_{23} \\ 7a_{31} & a_{32} & 2a_{33} \end{pmatrix}$$

Theorem Let $D = \mathrm{diag}(d_1, d_2, \ldots, d_n)$ and A be an $m \times n$ matrix. Then the matrix AD is obtained from A by multiplying the ith column by d_i for $i = 1, 2, \ldots, n$:

$$A = (\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n)$$
$$\implies \quad AD = (d_1\mathbf{w}_1, d_2\mathbf{w}_2, \ldots, d_n\mathbf{w}_n)$$

Identity matrix

Definition. The **identity matrix** (or **unit matrix**) is a diagonal matrix with all diagonal entries equal to 1.
The $n \times n$ identity matrix is denoted I_n or simply I.

$$I_1 = (1), \quad I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In general, $I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$.

Theorem. Let A be an arbitrary $m \times n$ matrix.
Then $I_m A = A I_n = A$.

Inverse matrix

Let $\mathcal{M}_n(\mathbb{R})$ denote the set of all $n \times n$ matrices with real entries. We can **add**, **subtract**, and **multiply** elements of $\mathcal{M}_n(\mathbb{R})$. What about **division**?

Definition. Let $A \in \mathcal{M}_n(\mathbb{R})$. Suppose there exists an $n \times n$ matrix B such that

$$AB = BA = I_n.$$

Then the matrix A is called **invertible** and B is called the **inverse** of A (denoted A^{-1}).

A non-invertible square matrix is called **singular**.

$$\boxed{AA^{-1} = A^{-1}A = I}$$

Examples

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$$AB = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$BA = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$C^2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus $A^{-1} = B$, $B^{-1} = A$, and $C^{-1} = C$.

Basic properties of inverse matrices:

- If $B = A^{-1}$ then $A = B^{-1}$. In other words, if A is invertible, so is A^{-1}, and $A = (A^{-1})^{-1}$.

- The inverse matrix (if it exists) is unique. Moreover, if $AB = CA = I$ for some matrices $B, C \in \mathcal{M}_n(\mathbb{R})$ then $B = C = A^{-1}$.

Indeed, $B = IB = (CA)B = C(AB) = CI = C$.

- If matrices $A, B \in \mathcal{M}_n(\mathbb{R})$ are invertible, so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.

$$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}IB = B^{-1}B = I,$$
$$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I.$$

- Similarly, $(A_1A_2 \dots A_k)^{-1} = A_k^{-1} \dots A_2^{-1}A_1^{-1}$.

Inverting diagonal matrices

Theorem A diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$ is invertible if and only if all diagonal entries are nonzero: $d_i \neq 0$ for $1 \leq i \leq n$.

If D is invertible then $D^{-1} = \mathrm{diag}(d_1^{-1}, \ldots, d_n^{-1})$.

$$\begin{pmatrix} d_1 & 0 & \ldots & 0 \\ 0 & d_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n \end{pmatrix}^{-1} = \begin{pmatrix} d_1^{-1} & 0 & \ldots & 0 \\ 0 & d_2^{-1} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & d_n^{-1} \end{pmatrix}$$

Inverting 2-by-2 matrices

Definition. The **determinant** of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\det A = ad - bc$.

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$.

If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Theorem A matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible if and only if $\det A \neq 0$. If $\det A \neq 0$ then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

Proof: Let $B = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Then

$$AB = BA = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = (ad - bc)I_2.$$

In the case $\det A \neq 0$, we have $A^{-1} = (\det A)^{-1}B$.
In the case $\det A = 0$, the matrix A is not invertible as otherwise $AB = O \implies A^{-1}AB = A^{-1}O \implies B = O \implies A = O$, but the zero matrix is singular.

Problem. Solve a system $\begin{cases} 4x + 3y = 5, \\ 3x + 2y = -1. \end{cases}$

This system is equivalent to a matrix equation $A\mathbf{x} = \mathbf{b}$,
where $A = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$.

We have $\det A = -1 \neq 0$. Hence A is invertible.

$A\mathbf{x} = \mathbf{b} \implies A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b} \implies (A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$
$\implies \mathbf{x} = A^{-1}\mathbf{b}$.

Conversely, $\mathbf{x} = A^{-1}\mathbf{b} \implies A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathbf{b}$.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \frac{1}{-1} \begin{pmatrix} 2 & -3 \\ -3 & 4 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \end{pmatrix} = \begin{pmatrix} -13 \\ 19 \end{pmatrix}$$

System of n linear equations in n variables:

$$\left\{\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{array}\right. \iff A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Theorem If the matrix A is invertible then the system has a unique solution, which is $\mathbf{x} = A^{-1}\mathbf{b}$.