

MATH 311

Topics in Applied Mathematics

**Lecture 6:
Evaluation of determinants.**

Determinant is a scalar assigned to each square matrix.

Notation. The determinant of a matrix

$A = (a_{ij})_{1 \leq i, j \leq n}$ is denoted $\det A$ or

$$\begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

Principal property: $\det A = 0$ if and only if the matrix A is singular.

Explicit definition in low dimensions

Definition. $\det(a) = a$, $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - \\ - a_{13}a_{22}a_{31} - a_{12}a_{21}a_{33} - a_{11}a_{23}a_{32}.$$

$$+ : \begin{pmatrix} \boxed{*} & * & * \\ * & \boxed{*} & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ * & * & \boxed{*} \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & * & \boxed{*} \\ \boxed{*} & * & * \\ * & \boxed{*} & * \end{pmatrix}.$$

$$- : \begin{pmatrix} * & * & \boxed{*} \\ * & \boxed{*} & * \\ \boxed{*} & * & * \end{pmatrix}, \begin{pmatrix} * & \boxed{*} & * \\ \boxed{*} & * & * \\ * & * & \boxed{*} \end{pmatrix}, \begin{pmatrix} \boxed{*} & * & * \\ * & * & \boxed{*} \\ * & \boxed{*} & * \end{pmatrix}.$$

Properties of determinants

Determinants and elementary row operations:

- if a row of a matrix is multiplied by a scalar r , the determinant is also multiplied by r ;
- if we add a row of a matrix multiplied by a scalar to another row, the determinant remains the same;
- if we interchange two rows of a matrix, the determinant changes its sign.

Properties of determinants

Tests for singularity:

- if a matrix A has a zero row then $\det A = 0$;
- if a matrix A has two identical rows then $\det A = 0$;
- if a matrix has two proportional rows then $\det A = 0$.

Properties of determinants

Special matrices:

- $\det I = 1$;
- the determinant of a diagonal matrix is equal to the product of its diagonal entries;
- the determinant of an upper triangular matrix is equal to the product of its diagonal entries.

Transpose of a matrix

Definition. Given a matrix A , the **transpose** of A , denoted A^T or A^t , is the matrix obtained by interchanging rows and columns in the matrix A . That is, if $A = (a_{ij})$ then $A^T = (b_{ij})$, where $b_{ij} = a_{ji}$.

Example.
$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}.$$

- If A is a square matrix then $\det A^T = \det A$.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Columns vs. rows

- If one column of a matrix is multiplied by a scalar, the determinant is multiplied by the same scalar.
- Interchanging two columns of a matrix changes the sign of its determinant.
- If a matrix A has two columns proportional then $\det A = 0$.
- Adding a scalar multiple of one column to another does not change the determinant of a matrix.

Submatrices

Definition. Given a matrix A , a $k \times k$ **submatrix** of A is a matrix obtained by specifying k columns and k rows of A and deleting the other columns and rows.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 10 & 20 & 30 & 40 \\ 3 & 5 & 7 & 9 \end{pmatrix} \rightarrow \begin{pmatrix} * & 2 & * & 4 \\ * & * & * & * \\ * & 5 & * & 9 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & 4 \\ 5 & 9 \end{pmatrix}$$

Given an $n \times n$ matrix A , let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the i th row and the j th column of A .

Example. $A = \begin{pmatrix} 3 & -2 & 0 \\ 1 & 0 & 1 \\ -2 & 3 & 0 \end{pmatrix}.$

$$M_{11} = \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix}, \quad M_{12} = \begin{pmatrix} 1 & 1 \\ -2 & 0 \end{pmatrix}, \quad M_{13} = \begin{pmatrix} 1 & 0 \\ -2 & 3 \end{pmatrix},$$

$$M_{21} = \begin{pmatrix} -2 & 0 \\ 3 & 0 \end{pmatrix}, \quad M_{22} = \begin{pmatrix} 3 & 0 \\ -2 & 0 \end{pmatrix}, \quad M_{23} = \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix},$$

$$M_{31} = \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \quad M_{32} = \begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}, \quad M_{33} = \begin{pmatrix} 3 & -2 \\ 1 & 0 \end{pmatrix}.$$

Row and column expansions

Given an $n \times n$ matrix $A = (a_{ij})$, let M_{ij} denote the $(n-1) \times (n-1)$ submatrix obtained by deleting the i th row and the j th column of A .

Theorem For any $1 \leq k, m \leq n$ we have that

$$\det A = \sum_{j=1}^n (-1)^{k+j} a_{kj} \det M_{kj},$$

(expansion by k th row)

$$\det A = \sum_{i=1}^n (-1)^{i+m} a_{im} \det M_{im}.$$

(expansion by m th column)

Signs for row/column expansions

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Expansion by the 1st row:

$$\begin{pmatrix} \boxed{1} & * & * \\ * & 5 & 6 \\ * & 8 & 9 \end{pmatrix} \quad \begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} * & * & \boxed{3} \\ 4 & 5 & * \\ 7 & 8 & * \end{pmatrix}$$

$$\begin{aligned} \det A &= 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= (5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0. \end{aligned}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Expansion by the 2nd column:

$$\begin{pmatrix} * & \boxed{2} & * \\ 4 & * & 6 \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ * & \boxed{5} & * \\ 7 & * & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & * & 3 \\ 4 & * & 6 \\ * & \boxed{8} & * \end{pmatrix}$$

$$\begin{aligned} \det A &= -2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} - 8 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2(4 \cdot 9 - 6 \cdot 7) + 5(1 \cdot 9 - 3 \cdot 7) - 8(1 \cdot 6 - 3 \cdot 4) = 0. \end{aligned}$$

Example. $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}.$

Subtract the 1st row from the 2nd row and from the 3rd row:

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \\ 6 & 6 & 6 \end{vmatrix} = 0$$

since the last matrix has two proportional rows.

Another example. $B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{pmatrix}$.

Let's do some row reduction.

Add -4 times the 1st row to the 2nd row:

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 13 \end{array} \right| = \left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{array} \right|$$

Add -7 times the 1st row to the 3rd row:

$$\left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{array} \right| = \left| \begin{array}{ccc} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{array} \right|$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 13 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix}$$

Expand the determinant by the 1st column:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -8 \end{vmatrix} = 1 \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix}$$

Thus

$$\begin{aligned} \det B &= \begin{vmatrix} -3 & -6 \\ -6 & -8 \end{vmatrix} = (-3) \begin{vmatrix} 1 & 2 \\ -6 & -8 \end{vmatrix} \\ &= (-3)(-2) \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = (-3)(-2)(-2) = -12 \end{aligned}$$

Example. $C = \begin{pmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{pmatrix}$, $\det C = ?$

Expand the determinant by the 3rd column:

$$\begin{vmatrix} 2 & -2 & 0 & 3 \\ -5 & 3 & 2 & 1 \\ 1 & -1 & 0 & -3 \\ 2 & 0 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Add -2 times the 2nd row to the 1st row:

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

$$\det C = -2 \begin{vmatrix} 2 & -2 & 3 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det C = -2 \begin{vmatrix} 0 & 0 & 9 \\ 1 & -1 & -3 \\ 2 & 0 & -1 \end{vmatrix} = -2 \cdot 9 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$$

Thus

$$\det C = -18 \begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix} = -18 \cdot 2 = -36$$

Problem. For what values of a will the following system have a unique solution?

$$\begin{cases} x + 2y + z = 1 \\ -x + 4y + 2z = 2 \\ 2x - 2y + az = 3 \end{cases}$$

The system has a unique solution if and only if the coefficient matrix is invertible.

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{pmatrix}, \quad \det A = ?$$

Add -2 times the 3rd column to the 2nd column:

$$\begin{vmatrix} 1 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -2 & a \end{vmatrix} = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2 - 2a & a \end{vmatrix}$$

Expand the determinant by the 2nd column:

$$\det A = \begin{vmatrix} 1 & 0 & 1 \\ -1 & 0 & 2 \\ 2 & -2 - 2a & a \end{vmatrix} = -(-2 - 2a) \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

Hence $\det A = -(-2 - 2a) \cdot 3 = 6(1 + a)$.

Thus A is invertible if and only if $a \neq -1$.

Properties of determinants

Determinants and matrix multiplication:

- if A and B are $n \times n$ matrices then
$$\det(AB) = \det A \cdot \det B;$$
- if A and B are $n \times n$ matrices then
$$\det(AB) = \det(BA);$$
- if A is an invertible matrix then
$$\det(A^{-1}) = (\det A)^{-1}.$$

Determinants and scalar multiplication:

- if A is an $n \times n$ matrix and $r \in \mathbb{R}$ then
$$\det(rA) = r^n \det A.$$

Examples

$$X = \begin{pmatrix} -1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & -3 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 2 & -2 & 1 \end{pmatrix}.$$

$$\det X = (-1) \cdot 2 \cdot (-3) = 6, \quad \det Y = \det Y^T = 3,$$

$$\det(XY) = 6 \cdot 3 = 18, \quad \det(YX) = 3 \cdot 6 = 18,$$

$$\det(Y^{-1}) = 1/3, \quad \det(XY^{-1}) = 6/3 = 2,$$

$$\det(XYX^{-1}) = \det Y = 3, \quad \det(X^{-1}Y^{-1}XY) = 1,$$

$$\det(2X) = 2^3 \det X = 2^3 \cdot 6 = 48,$$

$$\det(-3X^TXY^{-4}) = (-3)^3 \cdot 6 \cdot 6 \cdot 3^{-4} = -12.$$