MATH 311

Topics in Applied Mathematics

# Lecture 12:
# Change of coordinates (continued).
# Review for Test 1.

# Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space $V$, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

$$\textit{vector } \mathbf{v} \mapsto \textit{its coordinates } (x_1, x_2, \ldots, x_n)$$

is a one-to-one correspondence between $V$ and $\mathbb{R}^n$. This correspondence respects linear operations in $V$ and in $\mathbb{R}^n$.

# Change of coordinates in $\mathbb{R}^n$

The usual (standard) coordinates of a vector $\mathbf{v} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ are coordinates relative to the standard basis $\mathbf{e}_1 = (1, 0, \ldots, 0, 0)$, $\mathbf{e}_2 = (0, 1, \ldots, 0, 0)$, ..., $\mathbf{e}_n = (0, 0, \ldots, 0, 1)$.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for $\mathbb{R}^n$ and $(x'_1, x'_2, \ldots, x'_n)$ be the coordinates of the same vector $\mathbf{v}$ with respect to this basis.

**Problem 1.** Given the standard coordinates $(x_1, x_2, \ldots, x_n)$, find the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$.

**Problem 2.** Given the nonstandard coordinates $(x'_1, x'_2, \ldots, x'_n)$, find the standard coordinates $(x_1, x_2, \ldots, x_n)$.

It turns out that

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} u_{11} & u_{12} & \dots & u_{1n} \\ u_{21} & u_{22} & \dots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{n1} & u_{n2} & \dots & u_{nn} \end{pmatrix} \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}.$$

The matrix $U = (u_{ij})$ does not depend on the vector $\mathbf{x}$.

Columns of $U$ are coordinates of vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ with respect to the standard basis.

$U$ is called the **transition matrix** from the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ to the standard basis $\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n$.

This solves Problem 2. To solve Problem 1, we have to use the inverse matrix $U^{-1}$, which is the transition matrix from $\mathbf{e}_1, \dots, \mathbf{e}_n$ to $\mathbf{u}_1, \dots, \mathbf{u}_n$.

**Problem.** Find coordinates of the vector $\mathbf{x} = (1, 2, 3)$ with respect to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

The nonstandard coordinates $(x', y', z')$ of $\mathbf{x}$ satisfy

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = U \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

where $U$ is the transition matrix from the standard basis $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to the basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

The transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ is

$$U_0 = (\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3) = \left(\begin{array}{c|c|c} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{array}\right).$$

The transition matrix from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is the inverse matrix: $U = U_0^{-1}$.

The inverse matrix can be computed using row reduction.

$$(U_0 \mid I) = \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right) = (I \mid U_0^{-1})$$

Thus

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

# Change of coordinates: general case

Let $V$ be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis for $V$ and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$ be another basis for $V$ and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccccc}
 & & V & & \\
 & {\scriptstyle g_1}\swarrow & & \searrow{\scriptstyle g_2} & \\
\mathbb{R}^n & & \longrightarrow & & \mathbb{R}^n
\end{array}
$$

The composition $g_2 \circ g_1^{-1}$ is a transformation of $\mathbb{R}^n$.
It has the form $\mathbf{x} \mapsto U\mathbf{x}$, where $U$ is an $n \times n$ matrix.

$U$ is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \dots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \dots, \mathbf{u}_n$. Columns of $U$ are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n$.

**Problem.** Find the transition matrix from the basis $p_1(x) = 1$, $p_2(x) = x + 1$, $p_3(x) = (x + 1)^2$ to the basis $q_1(x) = 1$, $q_2(x) = x$, $q_3(x) = x^2$ for the vector space $\mathcal{P}_3$.

We have to find coordinates of the polynomials $p_1, p_2, p_3$ with respect to the basis $q_1, q_2, q_3$:

$p_1(x) = 1 = q_1(x)$,
$p_2(x) = x + 1 = q_1(x) + q_2(x)$,
$p_3(x) = (x+1)^2 = x^2+2x+1 = q_1(x)+2q_2(x)+q_3(x)$.

Thus the transition matrix is $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$.

**Problem.** Find the transition matrix from the basis $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (1, 0, 1)$, $\mathbf{v}_3 = (1, 2, 1)$ to the basis $\mathbf{u}_1 = (1, 1, 0)$, $\mathbf{u}_2 = (0, 1, 1)$, $\mathbf{u}_3 = (1, 1, 1)$.

It is convenient to make a two-step transition: first from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, and then from $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$.

Let $U_1$ be the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and $U_2$ be the transition matrix from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ to $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$:

$$U_1 = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then the transition matrix from $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ is $U_2^{-1}U_1$.

$$U_2^{-1}U_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$

## Topics for Test 1

*Part I: Elementary linear algebra (Leon 1.1–1.4, 2.1–2.2)*

- Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

- Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

- Matrix algebra. Inverse matrix.

- Determinants: explicit formulas for $2\times2$ and $3\times3$ matrices, row and column expansions, elementary row and column operations.

# Topics for Test 1

*Part II: Abstract linear algebra (Leon 3.1–3.4, 3.6)*

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.

# Sample problems for Test 1

**Problem 1 (15 pts.)** Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

**Problem 2 (25 pts.)** Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

(i) Evaluate the determinant of the matrix $A$.
(ii) Find the inverse matrix $A^{-1}$.

**Problem 3 (20 pts.)** Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

(i) The set $S_1$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.
(ii) The set $S_2$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.
(iii) The set $S_3$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.
(iv) The set $S_4$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

**Problem 4 (30 pts.)** Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

(i) Find the rank and the nullity of the matrix $B$.
(ii) Find a basis for the row space of $B$, then extend this basis to a basis for $\mathbb{R}^4$.
(iii) Find a basis for the nullspace of $B$.

**Bonus Problem 5 (15 pts.)** Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

**Bonus Problem 6 (15 pts.)** Let $V$ be a finite-dimensional vector space and $V_0$ be a proper subspace of $V$ (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

**Problem 1.** Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

The intersection point $(x, y, z)$ is a solution of the system

$$\begin{cases} x + 2y - z = 1, \\ x - 3y = -5, \\ 2x + y + z = 0. \end{cases}$$

To solve the system, we convert its augmented matrix into reduced row echelon form using elementary row operations:

$$\left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 1 & -3 & 0 & -5 \\ 2 & 1 & 1 & 0 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 2 & 1 & 1 & 0 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -5 & 1 & -6 \\ 0 & -3 & 3 & -2 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -3 & 3 & -2 \\ 0 & -5 & 1 & -6 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & -5 & 1 & -6 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & -4 & -\frac{8}{3} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -1 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & \frac{5}{3} \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & \frac{4}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right).$$

Thus the three planes intersect at the point $(-1, \frac{4}{3}, \frac{2}{3})$.

**Problem 1.** Find the point of intersection of the planes $x+2y-z=1$, $x-3y=-5$, and $2x+y+z=0$ in $\mathbb{R}^3$.

*Alternative solution:* The intersection point $(x, y, z)$ is a solution of the system

$$\begin{cases} x+2y-z=1, \\ x-3y=-5, \\ 2x+y+z=0. \end{cases}$$

Add all three equations: $4x=-4 \implies x=-1$.
Substitute $x=-1$ into the 2nd equation: $\implies y=\frac{4}{3}$.
Substitute $x=-1$ and $y=\frac{4}{3}$ into the 3rd equation:
$\implies z=\frac{2}{3}$.

It remains to check that $x=-1$, $y=\frac{4}{3}$, $z=\frac{2}{3}$ is indeed a solution of the system.

**Problem 2.** Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

**(i)** Evaluate the determinant of the matrix $A$.

Subtract the 4th row of $A$ from the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd row:

$$\begin{vmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 0 & 0 & -1 & 0 \\ 2 & 0 & 0 & 1 \end{vmatrix} = (-1)\begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix}.$$

Expand the determinant by the 3rd column:

$$(-1)\begin{vmatrix} 1 & -2 & 1 \\ 2 & 3 & 0 \\ 2 & 0 & 1 \end{vmatrix} = (-1)\left(\begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}\right) = -1.$$

**Problem 2.** Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

**(ii)** Find the inverse matrix $A^{-1}$.

First we merge the matrix $A$ with the identity matrix into one $4 \times 8$ matrix

$$(A \,|\, I) = \left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 2 & 3 & 2 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right).$$

Then we apply elementary row operations to this matrix until the left part becomes the identity matrix.

Subtract 2 times the 1st row from the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 3rd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 1st row from the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & 7 & -6 & -2 & -2 & 1 & 0 & 0 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract 2 times the 4th row from the 2nd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 4 & -9 & -1 & -2 & 0 & 1 & 0 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Subtract the 4th row from the 3rd row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 4 & -8 & -1 & -2 & 0 & 0 & 1 \end{array}\right)$$

Add 4 times the 2nd row to the 4th row:

$$\left(\begin{array}{cccc|cccc} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 32 & -1 & 6 & 4 & 0 & -7 \end{array}\right)$$

Add 32 times the 3rd row to the 4th row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 10 & 0 & 2 & 1 & 0 & -2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 10 times the 3rd row to the 2nd row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add the 4th row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 4 & 0 & 7 & 4 & 32 & -39 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Add 4 times the 3rd row to the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & -2 & 0 & 0 & 7 & 4 & 36 & -43 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Subtract 2 times the 2nd row from the 1st row:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & -1 & 0 & 0 & 2 & 1 & 10 & -12 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 6 & 4 & 32 & -39 \end{array}\right)$$

Multiply the 2nd, the 3rd, and the 4th rows by $-1$:

$$\left(\begin{array}{rrrr|rrrr} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right)$$

$$\left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 3 & 2 & 16 & -19 \\ 0 & 1 & 0 & 0 & -2 & -1 & -10 & 12 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & -6 & -4 & -32 & 39 \end{array}\right) = (I \mid A^{-1})$$

Finally the left part of our $4 \times 8$ matrix is transformed into the identity matrix. Therefore the current right part is the inverse matrix of $A$. Thus

$$A^{-1} = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 3 & 2 & 16 & -19 \\ -2 & -1 & -10 & 12 \\ 0 & 0 & -1 & 1 \\ -6 & -4 & -32 & 39 \end{pmatrix}.$$

**Problem 2.** Let $A = \begin{pmatrix} 1 & -2 & 4 & 1 \\ 2 & 3 & 2 & 0 \\ 2 & 0 & -1 & 1 \\ 2 & 0 & 0 & 1 \end{pmatrix}$.

**(i)** Evaluate the determinant of the matrix $A$.

*Alternative solution:* We have transformed $A$ into the identity matrix using elementary row operations. These included no row exchanges and three row multiplications, each time by $-1$.

It follows that $\det I = (-1)^3 \det A$.
$\implies \det A = -\det I = -1$.

**Problem 3.** Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

**(i)** The set $S_1$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $xyz = 0$.

$(0, 0, 0) \in S_1 \implies S_1$ is not empty.

$xyz = 0 \implies (rx)(ry)(rz) = r^3xyz = 0.$
That is, $\mathbf{v} = (x, y, z) \in S_1 \implies r\mathbf{v} = (rx, ry, rz) \in S_1.$
Hence $S_1$ is closed under scalar multiplication.

However $S_1$ is not closed under addition.
Counterexample: $(1, 1, 0) + (0, 0, 1) = (1, 1, 1).$

**Problem 3.** Determine which of the following subsets of $\mathbb{R}^3$ are subspaces. Briefly explain.

A subset of $\mathbb{R}^3$ is a subspace if it is closed under addition and scalar multiplication. Besides, the subset must not be empty.

**(ii)** The set $S_2$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $x + y + z = 0$.

$(0, 0, 0) \in S_2 \implies S_2$ is not empty.

$x + y + z = 0 \implies rx + ry + rz = r(x + y + z) = 0.$
Hence $S_2$ is closed under scalar multiplication.

$x + y + z = x' + y' + z' = 0 \implies$
$(x + x') + (y + y') + (z + z') = (x + y + z) + (x' + y' + z') = 0.$
That is, $\mathbf{v} = (x, y, z)$, $\mathbf{v}' = (x, y, z) \in S_2$
$\implies \mathbf{v} + \mathbf{v}' = (x + x', y + y', z + z') \in S_2.$
Hence $S_2$ is closed under addition.

**(iii)** The set $S_3$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 + z^2 = 0$.

$y^2 + z^2 = 0 \iff y = z = 0.$

$S_3$ is a nonempty set closed under addition and scalar multiplication.

**(iv)** The set $S_4$ of vectors $(x, y, z) \in \mathbb{R}^3$ such that $y^2 - z^2 = 0$.

$S_4$ is a nonempty set closed under scalar multiplication.
However $S_4$ is not closed under addition.
Counterexample: $(0, 1, 1) + (0, 1, -1) = (0, 2, 0)$.

**Problem 4.** Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(i)** Find the rank and the nullity of the matrix $B$.

The rank (= dimension of the row space) and the nullity (= dimension of the nullspace) of a matrix are preserved under elementary row operations. We apply such operations to convert the matrix $B$ into row echelon form.

Interchange the 1st row with the 2nd row:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$$

Add 3 times the 1st row to the 3rd row, then subtract 2 times the 1st row from the 4th row:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & -1 & 4 & 1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Multiply the 2nd row by $-1$:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 3 & 5 & -3 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add the 4th row to the 3rd row:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -3 & -4 & 3 \end{pmatrix}$$

Add 3 times the 2nd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -16 & 0 \end{pmatrix}$$

Add 16 times the 3rd row to the 4th row:

$$\rightarrow \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Now that the matrix is in row echelon form, its rank equals the number of nonzero rows, which is 3. Since

$$(\text{rank of } B) + (\text{nullity of } B) = (\text{the number of columns of } B) = 4,$$

it follows that the nullity of $B$ equals 1.

**Problem 4.** Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(ii)** Find a basis for the row space of $B$, then extend this basis to a basis for $\mathbb{R}^4$.

The row space of a matrix is invariant under elementary row operations. Therefore the row space of the matrix $B$ is the same as the row space of its row echelon form:

$$\begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The nonzero rows of the latter matrix are linearly independent so that they form a basis for its row space:

$$\mathbf{v}_1 = (1, 1, 2, -1), \;\; \mathbf{v}_2 = (0, 1, -4, -1), \;\; \mathbf{v}_3 = (0, 0, 1, 0).$$

To extend the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ to a basis for $\mathbb{R}^4$, we need a vector $\mathbf{v}_4 \in \mathbb{R}^4$ that is not a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

It is known that at least one of the vectors $\mathbf{e}_1 = (1, 0, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0, 0)$, $\mathbf{e}_3 = (0, 0, 1, 0)$, and $\mathbf{e}_4 = (0, 0, 0, 1)$ can be chosen as $\mathbf{v}_4$.

In particular, the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{e}_4$ form a basis for $\mathbb{R}^4$. This follows from the fact that the $4 \times 4$ matrix whose rows are these vectors is not singular:

$$\begin{vmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 1 \neq 0.$$

**Problem 4.** Let $B = \begin{pmatrix} 0 & -1 & 4 & 1 \\ 1 & 1 & 2 & -1 \\ -3 & 0 & -1 & 0 \\ 2 & -1 & 0 & 1 \end{pmatrix}$.

**(iii)** Find a basis for the nullspace of $B$.

The nullspace of $B$ is the solution set of the system of linear homogeneous equations with $B$ as the coefficient matrix. To solve the system, we convert $B$ to reduced row echelon form:

$$\to \begin{pmatrix} 1 & 1 & 2 & -1 \\ 0 & 1 & -4 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\implies \quad x_1 = x_2 - x_4 = x_3 = 0$

General solution: $(x_1, x_2, x_3, x_4) = (0, t, 0, t) = t(0, 1, 0, 1)$.

Thus the vector $(0, 1, 0, 1)$ forms a basis for the nullspace of $B$.

**Bonus Problem 5.** Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

Suppose that $af_1(x) + bf_2(x) + cf_3(x) = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

Let us differentiate the identity 4 times:

$$\begin{aligned} ax + bxe^x + ce^{-x} &= 0, \\ a + be^x + bxe^x - ce^{-x} &= 0, \\ 2be^x + bxe^x + ce^{-x} &= 0, \\ 3be^x + bxe^x - ce^{-x} &= 0, \\ 4be^x + bxe^x + ce^{-x} &= 0. \end{aligned}$$

(the 5th identity)−(the 3rd identity): $2be^x = 0 \implies b = 0$.

Substitute $b = 0$ in the 3rd identity: $ce^{-x} = 0 \implies c = 0$.

Substitute $b = c = 0$ in the 2nd identity: $a = 0$.

**Bonus Problem 5.** Show that the functions $f_1(x) = x$, $f_2(x) = xe^x$, and $f_3(x) = e^{-x}$ are linearly independent in the vector space $C^\infty(\mathbb{R})$.

*Alternative solution:* Suppose that $ax + bxe^x + ce^{-x} = 0$ for all $x \in \mathbb{R}$, where $a, b, c$ are constants. We have to show that $a = b = c = 0$.

For any $x \neq 0$ divide both sides of the identity by $xe^x$:

$$ae^{-x} + b + cx^{-1}e^{-2x} = 0.$$

The left-hand side approaches $b$ as $x \to +\infty$. $\implies b = 0$

Now $ax + ce^{-x} = 0$ for all $x \in \mathbb{R}$. For any $x \neq 0$ divide both sides of the identity by $x$:

$$a + cx^{-1}e^{-x} = 0.$$

The left-hand side approaches $a$ as $x \to +\infty$. $\implies a = 0$

Now $ce^{-x} = 0 \implies c = 0$.

**Bonus Problem 6.** Let $V$ be a finite-dimensional vector space and $V_0$ be a proper subspace of $V$ (where proper means that $V_0 \neq V$). Prove that $\dim V_0 < \dim V$.

Any vector space has a basis. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be a basis for $V_0$.

Vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly independent in $V$ since they are linearly independent in $V_0$. Therefore we can extend this collection of vectors to a basis for $V$ by adding some vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$. As $V_0 \neq V$, we do need to add some vectors, i.e., $m \geq 1$.

Thus $\dim V_0 = k$ and $\dim V = k + m > k$.