MATH 311

Topics in Applied Mathematics

Lecture 14a:
Matrix of a linear transformation.

Linear transformation

Definition. Given vector spaces V_1 and V_2, a mapping $L : V_1 \to V_2$ is **linear** if

$$\boxed{L(\mathbf{x} + \mathbf{y}) = L(\mathbf{x}) + L(\mathbf{y}),}$$
$$\boxed{L(r\mathbf{x}) = rL(\mathbf{x})}$$

for any $\mathbf{x}, \mathbf{y} \in V_1$ and $r \in \mathbb{R}$.

Basic properties of linear mappings:

• $L(r_1\mathbf{v}_1 + \cdots + r_k\mathbf{v}_k) = r_1 L(\mathbf{v}_1) + \cdots + r_k L(\mathbf{v}_k)$
for all $k \geq 1$, $\mathbf{v}_1, \ldots, \mathbf{v}_k \in V_1$, and $r_1, \ldots, r_k \in \mathbb{R}$.

• $L(\mathbf{0}_1) = \mathbf{0}_2$, where $\mathbf{0}_1$ and $\mathbf{0}_2$ are zero vectors in V_1 and V_2, respectively.

• $L(-\mathbf{v}) = -L(\mathbf{v})$ for any $\mathbf{v} \in V_1$.

Matrix transformations

Theorem Suppose $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map. Then there exists an $m \times n$ matrix A such that $L(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$. Columns of A are vectors $L(\mathbf{e}_1), L(\mathbf{e}_2), \ldots, L(\mathbf{e}_n)$, where $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ is the standard basis for $\mathbb{R}^n$.

$$
\mathbf{y} = A\mathbf{x} \quad \Longleftrightarrow \quad
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}
=
\begin{pmatrix}
a_{11} & a_{12} & \ldots & a_{1n} \\
a_{21} & a_{22} & \ldots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & \ldots & a_{mn}
\end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
$$

$$
\Longleftrightarrow \quad
\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}
= x_1 \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}
+ x_2 \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}
+ \cdots + x_n \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}
$$

Basis and coordinates

If $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is a basis for a vector space V, then any vector $\mathbf{v} \in V$ has a unique representation

$$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n,$$

where $x_i \in \mathbb{R}$. The coefficients $x_1, x_2, \ldots, x_n$ are called the **coordinates** of $\mathbf{v}$ with respect to the ordered basis $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.

The mapping

vector $\mathbf{v} \mapsto$ *its coordinates* $(x_1, x_2, \ldots, x_n)$

provides a one-to-one correspondence between V and $\mathbb{R}^n$. Besides, this mapping is **linear**.

Change of coordinates

Let V be a vector space.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ be another basis for V and $g_2 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccc}
 & V & \\
{\scriptstyle g_1}\swarrow & & \searrow{\scriptstyle g_2} \\
\mathbb{R}^n & \longrightarrow & \mathbb{R}^n
\end{array}
$$
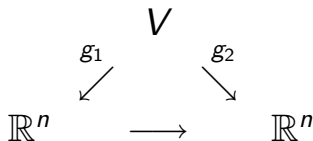
The composition $g_2 \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to itself. It is represented as $\mathbf{x} \mapsto U\mathbf{x}$, where U is an $n \times n$ matrix.

U is called the **transition matrix** from $\mathbf{v}_1, \mathbf{v}_2 \ldots, \mathbf{v}_n$ to $\mathbf{u}_1, \mathbf{u}_2 \ldots, \mathbf{u}_n$. Columns of U are coordinates of the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ with respect to the basis $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$.

Matrix of a linear transformation

Let V, W be vector spaces and $f : V \to W$ be a linear map.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a basis for V and $g_1 : V \to \mathbb{R}^n$ be the coordinate mapping corresponding to this basis.

Let $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$ be a basis for W and $g_2 : W \to \mathbb{R}^m$ be the coordinate mapping corresponding to this basis.

$$
\begin{array}{ccc}
V & \xrightarrow{\;f\;} & W \\
g_1 \downarrow & & \downarrow g_2 \\
\mathbb{R}^n & \longrightarrow & \mathbb{R}^m
\end{array}
$$

The composition $g_2 \circ f \circ g_1^{-1}$ is a linear mapping of $\mathbb{R}^n$ to $\mathbb{R}^m$. It is represented as $\mathbf{x} \mapsto A\mathbf{x}$, where A is an $m \times n$ matrix.

A is called the **matrix of** f with respect to bases $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and $\mathbf{w}_1, \ldots, \mathbf{w}_m$. Columns of A are coordinates of vectors $f(\mathbf{v}_1), \ldots, f(\mathbf{v}_n)$ with respect to the basis $\mathbf{w}_1, \ldots, \mathbf{w}_m$.

Examples. • $D : \mathcal{P}_3 \to \mathcal{P}_2, \ (Dp)(x) = p'(x)$.

Let A_D be the matrix of D with respect to the bases $1, x, x^2$ and $1, x$. Columns of A_D are coordinates of polynomials $D1, \ Dx, \ Dx^2$ w.r.t. the basis $1, x$.

$$D1 = 0, \ Dx = 1, \ Dx^2 = 2x \implies A_D = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

• $L : \mathcal{P}_3 \to \mathcal{P}_3, \ (Lp)(x) = p(x+1)$.

Let A_L be the matrix of L w.r.t. the basis $1, x, x^2$.
$L1 = 1, \ Lx = 1 + x, \ Lx^2 = (x+1)^2 = 1 + 2x + x^2$.

$$\implies A_L = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$,

$$L \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis
$\mathbf{v}_1 = (3, 1), \ \mathbf{v}_2 = (2, 1)$.

Let N be the desired matrix. Columns of N are coordinates of
the vectors $L(\mathbf{v}_1)$ and $L(\mathbf{v}_2)$ w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2$.

$$L(\mathbf{v}_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}, \quad L(\mathbf{v}_2) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}.$$

Clearly, $L(\mathbf{v}_2) = \mathbf{v}_1 = 1\mathbf{v}_1 + 0\mathbf{v}_2$.

$$L(\mathbf{v}_1) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 \iff \begin{cases} 3\alpha + 2\beta = 4 \\ \alpha + \beta = 1 \end{cases} \iff \begin{cases} \alpha = 2 \\ \beta = -1 \end{cases}$$

Thus $N = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$.

Change of basis for a linear operator

Let $L : V \to V$ be a linear operator on a vector space V.

Let A be the matrix of L relative to a basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ for V. Let B be the matrix of L relative to another basis $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$ for V.

Let U be the transition matrix from the basis $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ to $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_n$.

$$
\begin{array}{ccc}
\boxed{\mathbf{a}\text{-coordinates of } \mathbf{v}} & \xrightarrow{\ A\ } & \boxed{\mathbf{a}\text{-coordinates of } L(\mathbf{v})} \\
U \downarrow & & \downarrow U \\
\boxed{\mathbf{b}\text{-coordinates of } \mathbf{v}} & \xrightarrow{\ B\ } & \boxed{\mathbf{b}\text{-coordinates of } L(\mathbf{v})}
\end{array}
$$

It follows that $UA = BU$.

Then $A = U^{-1}BU$ and $B = UAU^{-1}$.

Problem. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$,
$$L\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

Find the matrix of L with respect to the basis
$\mathbf{v}_1 = (3, 1)$, $\mathbf{v}_2 = (2, 1)$.

Let S be the matrix of L with respect to the standard basis,
N be the matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$, and U be
the transition matrix from $\mathbf{v}_1, \mathbf{v}_2$ to $\mathbf{e}_1, \mathbf{e}_2$. Then $N = U^{-1}SU$.

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad U = \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix},$$

$$N = U^{-1}SU = \begin{pmatrix} 1 & -2 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

Linear transformations of $\mathbb{R}^2$

Any linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ is represented as multiplication of a 2-dimensional column vector by a 2×2 matrix: $f(\mathbf{x}) = A\mathbf{x}$ or

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Linear transformations corresponding to particular matrices can have various geometric properties.
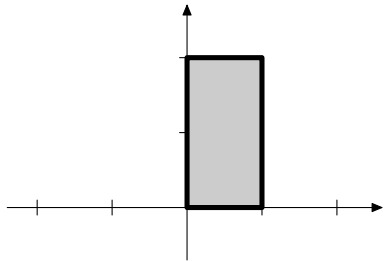
$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$
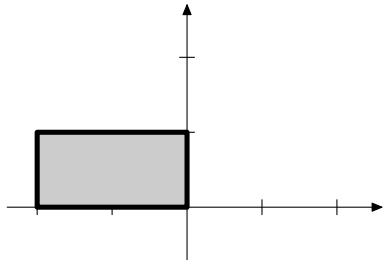
Rotation by 90°
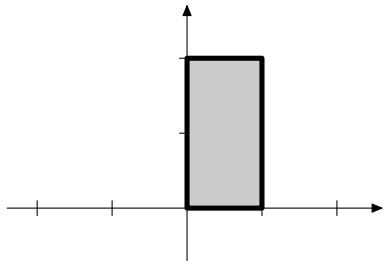
$$A = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$
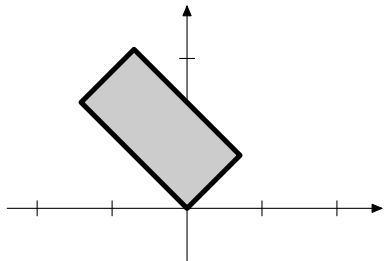
Rotation by 45°
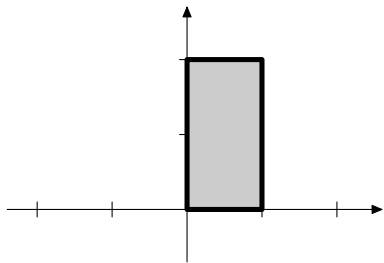
$$A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$
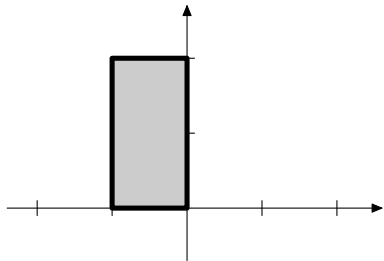
Reflection in
the vertical axis
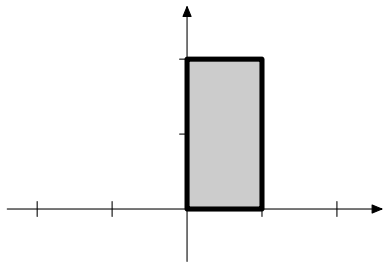
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
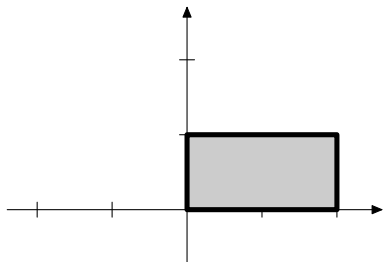
Reflection in
the line $x - y = 0$
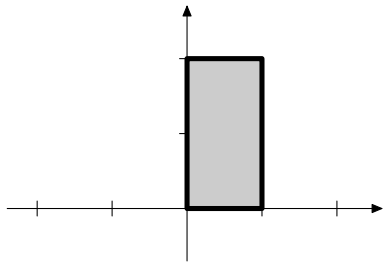
$$A = \begin{pmatrix} 1 & 1/2 \\ 0 & 1 \end{pmatrix}$$
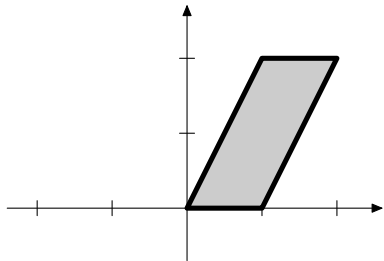
Horizontal shear
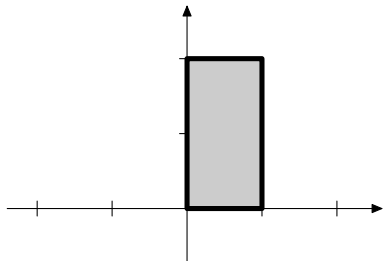
$$A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}$$
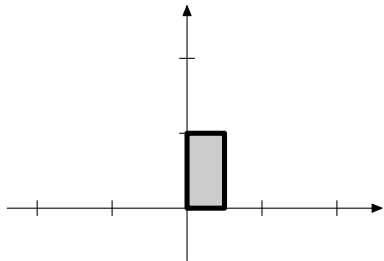
Scaling
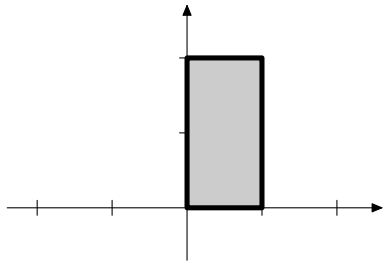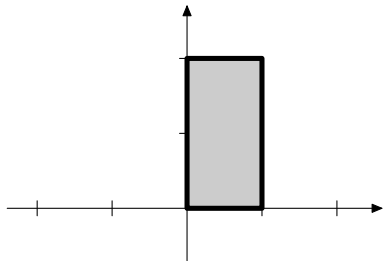
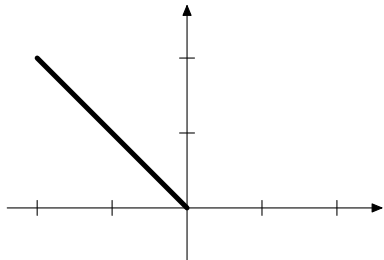$$A = \begin{pmatrix} 3 & 0 \\ 0 & 1/3 \end{pmatrix}$$
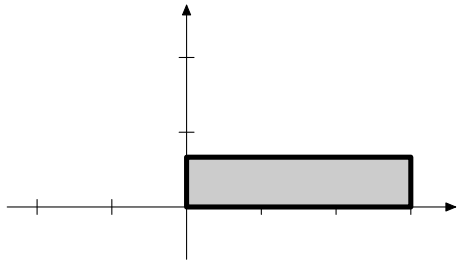
Squeeze
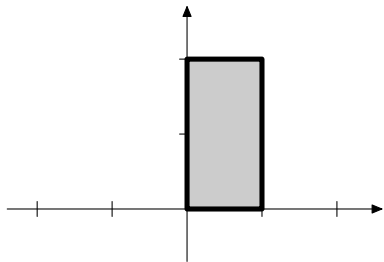
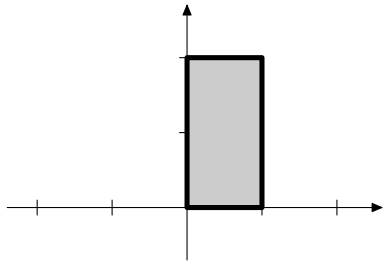$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
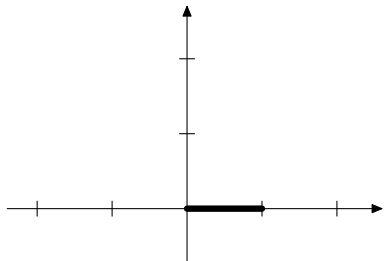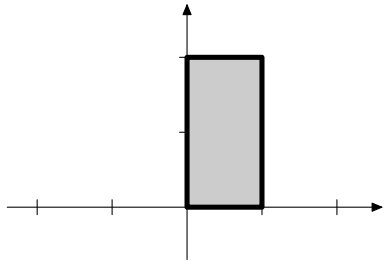
Vertical projection on the horizontal axis

$$A = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$$

Horizontal projection
on the line $x + y = 0$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Identity