MATH 311

Topics in Applied Mathematics

Lecture 14b: Eigenvalues and eigenvectors.

Eigenvalues and eigenvectors

Definition. Let A be an $n \times n$ matrix. A number $\lambda \in \mathbb{R}$ is called an **eigenvalue** of the matrix A if $\boxed{A\mathbf{v} = \lambda\mathbf{v}}$ for a nonzero column vector $\mathbf{v} \in \mathbb{R}^n$.

The vector $\mathbf{v}$ is called an **eigenvector** of A belonging to (or associated with) the eigenvalue λ.

Remarks. • Alternative notation:
eigenvalue = **characteristic value**,
eigenvector = **characteristic vector**.

• The zero vector is never considered an eigenvector.

Example. $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Hence $(1, 0)$ is an eigenvector of A belonging to the eigenvalue 2, while $(0, -2)$ is an eigenvector of A belonging to the eigenvalue 3.

Example. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Hence $(1, 1)$ is an eigenvector of A belonging to the eigenvalue 1, while $(1, -1)$ is an eigenvector of A belonging to the eigenvalue -1.

Vectors $\mathbf{v}_1 = (1, 1)$ and $\mathbf{v}_2 = (1, -1)$ form a basis for $\mathbb{R}^2$. Consider a linear operator $L : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L(\mathbf{x}) = A\mathbf{x}$. The matrix of L with respect to the basis $\mathbf{v}_1, \mathbf{v}_2$ is $B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Let A be an $n \times n$ matrix. Consider a linear operator $L : \mathbb{R}^n \to \mathbb{R}^n$ given by $L(\mathbf{x}) = A\mathbf{x}$.

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be a nonstandard basis for $\mathbb{R}^n$ and B be the matrix of the operator L with respect to this basis.

Theorem The matrix B is diagonal if and only if vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are eigenvectors of A.

If this is the case, then the diagonal entries of the matrix B are the corresponding eigenvalues of A.

$$A\mathbf{v}_i = \lambda_i \mathbf{v}_i \iff B = \begin{pmatrix} \lambda_1 & & & O \\ & \lambda_2 & & \\ & & \ddots & \\ O & & & \lambda_n \end{pmatrix}$$

Eigenspaces

Let A be an $n \times n$ matrix. Let $\mathbf{v}$ be an eigenvector of A belonging to an eigenvalue λ.

Then $A\mathbf{v} = \lambda\mathbf{v} \implies A\mathbf{v} = (\lambda I)\mathbf{v} \implies (A - \lambda I)\mathbf{v} = \mathbf{0}$.

Hence $\mathbf{v} \in N(A - \lambda I)$, the nullspace of the matrix $A - \lambda I$.

Conversely, if $\mathbf{x} \in N(A - \lambda I)$ then $A\mathbf{x} = \lambda\mathbf{x}$. Thus the eigenvectors of A belonging to the eigenvalue λ are nonzero vectors from $N(A - \lambda I)$.

Definition. If $N(A - \lambda I) \neq \{\mathbf{0}\}$ then it is called the **eigenspace** of the matrix A corresponding to the eigenvalue λ.

How to find eigenvalues and eigenvectors?

Theorem Given a square matrix A and a scalar λ, the following statements are equivalent:

- λ is an eigenvalue of A,
- $N(A - \lambda I) \neq \{\mathbf{0}\}$,
- the matrix $A - \lambda I$ is singular,
- $\det(A - \lambda I) = 0$.

Definition. $\det(A - \lambda I) = 0$ is called the **characteristic equation** of the matrix A.

Eigenvalues λ of A are roots of the characteristic equation. Associated eigenvectors of A are nonzero solutions of the equation $(A - \lambda I)\mathbf{x} = \mathbf{0}$.