MATH 311

Topics in Applied Mathematics

Lecture 21:
Boundary value problems.
Separation of variables.

Differential equations

A **differential equation** is an equation involving an unknown function and certain of its derivatives.

An **ordinary differential equation (ODE)** is an equation involving an unknown function of one variable and certain of its derivatives.

A **partial differential equation (PDE)** is an equation involving an unknown function of two or more variables and certain of its partial derivatives.

Examples

$$x^2 + 2x + 1 = 0 \qquad \text{(algebraic equation)}$$

$$f(2x) = 2(f(x))^2 - 1 \qquad \text{(functional equation)}$$

$$f'(t) + t^2 f(t) = 4 \qquad \text{(ODE)}$$

$$\frac{\partial u}{\partial x} + 3\frac{\partial^2 u}{\partial x \partial y} - u\frac{\partial u}{\partial y} \qquad \text{(not an equation)}$$

$$\frac{\partial u}{\partial x} - 5\frac{\partial u}{\partial y} = u \qquad \text{(PDE)}$$

$$u + u^2 = \frac{\partial^2 u}{\partial x \partial y}(0,0) \qquad \text{(functional-differential equation)}$$

heat equation: $$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

In the first two equations, $u = u(x, t)$. In the latter one, $u = u(x, y)$.

heat equation: $$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

In the first two equations, $u = u(x, y, t)$. In the latter one, $u = u(x, y, z)$.

Initial and boundary conditions for ODEs

$y'(t) = y(t)$, $0 \leq t \leq L$.

General solution: $y(t) = C_1 e^t$, where $C_1 = \text{const}$.

To determine a unique solution, we need one **initial condition**.

For example, $y(0) = 1$. Then $y(t) = e^t$ is the unique solution.

$y''(t) = -y(t)$, $0 \le t \le L$.
General solution: $y(t) = C_1 \cos t + C_2 \sin t$, where C_1, C_2 are constant.

To determine a unique solution, we need **two** initial conditions. For example, $y(0) = 1$, $y'(0) = 0$. Then $y(t) = \cos t$ is the unique solution.

Alternatively, we may impose boundary conditions. For example, $y(0) = 0$, $y(L) = 1$. In the case $L = \pi/2$, $y(t) = \sin t$ is the unique solution.

PDE

$$\frac{\partial^2 u}{\partial w \, \partial z} = 0, \quad u = u(w, z)$$

Domain: $a_1 \leq w \leq a_2$, $b_1 \leq z \leq b_2$.

(we allow intervals $[a_1, a_2]$ and $[b_1, b_2]$ to be infinite or semi-infinite)

$$\frac{\partial}{\partial w}\left(\frac{\partial u}{\partial z}\right) = 0, \qquad \frac{\partial u}{\partial z}(w, z) = \gamma(z)$$

$$u(w, z) = \int_{z_0}^{z} \gamma(\xi) \, d\xi + C(w)$$

$$\boxed{u(w, z) = B(z) + C(w)} \quad \text{(general solution)}$$

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Change of independent variables:

$$w = x + ct, \qquad z = x - ct.$$

How does the equation look in new coordinates?

$$\frac{\partial}{\partial t} = \frac{\partial w}{\partial t} \frac{\partial}{\partial w} + \frac{\partial z}{\partial t} \frac{\partial}{\partial z} = c \frac{\partial}{\partial w} - c \frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} = \frac{\partial w}{\partial x} \frac{\partial}{\partial w} + \frac{\partial z}{\partial x} \frac{\partial}{\partial z} = \frac{\partial}{\partial w} + \frac{\partial}{\partial z}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}\right) u$$

$$= c^2 \left(\frac{\partial^2 u}{\partial w^2} - 2\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}\right).$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial w^2} + 2\frac{\partial^2 u}{\partial w\,\partial z} + \frac{\partial^2 u}{\partial z^2}.$$

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = -4c^2\frac{\partial^2 u}{\partial w\,\partial z}.$$

Wave equation in new coordinates: $\dfrac{\partial^2 u}{\partial w\,\partial z} = 0.$

General solution: $\boxed{u(x,t) = B(x-ct) + C(x+ct)}$

(d'Alembert, 1747)

Boundary conditions for PDEs

Heat equation: $\dfrac{\partial u}{\partial t} = k\,\dfrac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,$
$0 \le t \le T$.

Initial condition: $u(x,0) = f(x)$, where
$f : [0, L] \to \mathbb{R}$.

Boundary conditions: $u(0,t) = u_1(t)$,
$u(L,t) = u_2(t)$, where $u_1, u_2 : [0, T] \to \mathbb{R}$.

Boundary conditions of the **first kind**: prescribed temperature.

Another boundary conditions: $\dfrac{\partial u}{\partial x}(0, t) = \phi_1(t)$, $\dfrac{\partial u}{\partial x}(L, t) = \phi_2(t)$, where $\phi_1, \phi_2 : [0, T] \to \mathbb{R}$.

Boundary conditions of the **second kind**: prescribed heat flux.

A particular case: $\dfrac{\partial u}{\partial x}(0, t) = \dfrac{\partial u}{\partial x}(L, t) = 0$ (insulated boundary).

Robin conditions:

$$-\frac{\partial u}{\partial x}(0,t) = -h \cdot \Big(u(0,t) - u_1(t)\Big),$$

$$-\frac{\partial u}{\partial x}(L,t) = h \cdot \Big(u(L,t) - u_2(t)\Big),$$

where $h = \text{const} > 0$ and $u_1, u_2 : [0, T] \to \mathbb{R}$.

Boundary conditions of the **third kind**: Newton's law of cooling.

Also, we may consider **mixed** boundary conditions, for example, $u(0,t) = u_1(t)$, $\dfrac{\partial u}{\partial x}(L,t) = \phi_2(t)$.

Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 \le x \le L,\ 0 \le t \le T.$$

Two initial conditions: $u(x, 0) = f(x)$,
$\frac{\partial u}{\partial t}(x, 0) = g(x)$, where $f, g : [0, L] \to \mathbb{R}$.

Some boundary conditions: $u(0, t) = u(L, t) = 0$.

Dirichlet conditions: fixed ends.

Another boundary conditions:
$\frac{\partial u}{\partial x}(0, t) = \frac{\partial u}{\partial x}(L, t) = 0$.

Neumann conditions: free ends.

Linear equations

An equation is called **linear** if it can be written in the form

$$L(u) = f,$$

where $L : V_1 \to V_2$ is a linear map, $f \in V_2$ is given, and $u \in V_1$ is the unknown. If $f = 0$ then the linear equation is called **homogeneous**.

Theorem The general solution of a linear equation $L(u) = f$ is

$$u = u_1 + u_0,$$

where u_1 is a particular solution and u_0 is the general solution of the homogeneous equation $L(u) = 0$.

Linear differential operators

- ordinary differential operator:

$$L = g_0 \frac{d^2}{dx^2} + g_1 \frac{d}{dx} + g_2 \quad (g_0, g_1, g_2 \text{ are functions})$$

- heat operator: $L = \dfrac{\partial}{\partial t} - k\,\dfrac{\partial^2}{\partial x^2}$

- wave operator: $L = \dfrac{\partial^2}{\partial t^2} - c^2\,\dfrac{\partial^2}{\partial x^2}$

(a.k.a. the d'Alembertian; denoted by $\square$).

- Laplace's operator: $L = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$

(a.k.a. the Laplacian; denoted by Δ or ∇^2).

How do we solve a linear homogeneous PDE?

Step 1: Find some solutions.

Step 2: Form linear combinations of solutions obtained on Step 1.

Step 3: Show that every solution can be approximated by solutions obtained on Step 2.

Similarly, we solve a linear homogeneous PDE with linear homogeneous boundary conditions (boundary problem).

One way to complete Step 1: the method of **separation of variables**.

Separation of variables

The method applies to certain linear PDEs. It is used to find some solutions.

Basic idea: to find a solution of the PDE (function of many variables) as a combination of several functions, each depending only on one variable.

For example, $u(x,t) = B(x) + C(t)$ or $u(x,t) = B(x)C(t)$.

The first example works perfectly for one equation: $\frac{\partial^2 u}{\partial t\,\partial x} = 0$.

The second example proved useful for **many** equations.

Heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

Suppose $u(x,t) = \phi(x)G(t)$. Then

$$\frac{\partial u}{\partial t} = \phi(x)\frac{dG}{dt}, \qquad \frac{\partial^2 u}{\partial x^2} = \frac{d^2\phi}{dx^2}G(t).$$

Hence

$$\phi(x)\frac{dG}{dt} = k\,\frac{d^2\phi}{dx^2}G(t).$$

Divide both sides by $k \cdot \phi(x) \cdot G(t) = k \cdot u(x,t)$:

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2}.$$

It follows that

$$\frac{1}{kG} \cdot \frac{dG}{dt} = \frac{1}{\phi} \cdot \frac{d^2\phi}{dx^2} = -\lambda = \text{const.}$$

λ is called the **separation constant**. The variables have been separated:

$$\begin{gathered} \tfrac{d^2\phi}{dx^2} = -\lambda\phi, \\ \tfrac{dG}{dt} = -\lambda k G. \end{gathered}$$

Proposition Suppose ϕ and G are solutions of the above ODEs for the same value of λ. Then $u(x,t) = \phi(x)G(t)$ is a solution of the heat equation.

Example. $u(x,t) = e^{-kt}\sin x$.

$$\boxed{\frac{dG}{dt} = -\lambda k G}$$

General solution: $G(t) = C_0 e^{-\lambda k t}$, $C_0 = \text{const.}$

$$\boxed{\frac{d^2\phi}{dx^2} = -\lambda \phi}$$

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

Case 1: $\lambda > 0$. Then $\lambda = \mu^2$, where $\mu > 0$.
$\phi(x) = C_1 \cos \mu x + C_2 \sin \mu x, \quad C_1, C_2 = \text{const.}$

Case 2: $\lambda = 0$. $\phi(x) = C_1 + C_2 x$.

Case 3: $\lambda < 0$. Then $\lambda = -\mu^2$, where $\mu > 0$.
$\phi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$.

Theorem For any $C_1, C_2 \in \mathbb{R}$ and $\mu > 0$, the functions

$$\begin{aligned} u_+(x,t) &= e^{-k\mu^2 t}(C_1 \cos \mu x + C_2 \sin \mu x), \\ u_0(x,t) &= C_1 + C_2 x, \\ u_-(x,t) &= e^{k\mu^2 t}(C_1 e^{\mu x} + C_2 e^{-\mu x}) \end{aligned}$$

are solutions of the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}.$$

Laplace's equation inside a rectangle

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < L,\ 0 < y < H)$$

Boundary conditions:

$$\begin{aligned} u(0, y) &= g_1(y) \\ u(L, y) &= g_2(y) \\ u(x, 0) &= f_1(x) \\ u(x, H) &= f_2(x) \end{aligned}$$

Principle of superposition:

$$u = u_1 + u_2 + u_3 + u_4,$$

where

$$\nabla^2 u_1 = \nabla^2 u_2 = \nabla^2 u_3 = \nabla^2 u_4 = 0,$$

$$u_1(x,0) = f_1(x), \;\; u_1(0,y) = u_1(L,y) = u_1(x,H) = 0;$$
$$u_2(L,y) = g_2(y), \;\; u_2(0,y) = u_2(x,0) = u_2(x,H) = 0;$$
$$u_3(x,H) = f_2(x), \;\; u_3(0,y) = u_3(L,y) = u_3(x,0) = 0;$$
$$u_4(0,y) = g_1(y), \;\; u_4(L,y) = u_4(x,0) = u_4(x,H) = 0.$$

Reduced boundary value problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad (0 < x < L,\ 0 < y < H)$$

Boundary conditions:

$$\begin{aligned} u(0, y) &= 0 \\ u(L, y) &= 0 \\ u(x, 0) &= f_1(x) \\ u(x, H) &= 0 \end{aligned}$$

Separation of variables

We are looking for a solution $u(x, y) = \phi(x)h(y)$ satisfying all 3 homogeneous boundary conditions.

(Next step will be to combine such solutions into one that satisfies the nonhomogeneous boundary condition as well.)

PDE holds if

$$\frac{d^2\phi}{dx^2} = -\lambda\phi \quad \text{and} \quad \frac{d^2h}{dy^2} = \lambda h$$

for the same constant λ.

Boundary conditions $u(0, y) = u(L, y) = 0$ hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary condition $u(x, H) = 0$ holds if

$$h(H) = 0.$$

Eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}$.

Dependence on y:

$$h'' = \lambda h, \qquad h(H) = 0.$$

$$\implies h(y) = C_0 \sinh\sqrt{\lambda}(y - H)$$

Solution of Laplace's equation:

$$u(x, y) = \sin\frac{n\pi x}{L}\sinh\frac{n\pi(y-H)}{L}, \qquad n = 1, 2, \dots$$