MATH 311

Topics in Applied Mathematics

Lecture 22:
Fourier's solution of the heat equation.
Fourier series.

PDEs: two variables

heat equation: $$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}$$

wave equation: $$\frac{\partial^2 u}{\partial t^2} = c^2\,\frac{\partial^2 u}{\partial x^2}$$

Laplace's equation: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

These equations are **linear homogeneous**.

One-dimensional heat equation

Describes heat conduction in a rod:

$$\boxed{c\rho\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_0\frac{\partial u}{\partial x}\right) + Q}$$

$K_0 = K_0(x)$, $c = c(x)$, $\rho = \rho(x)$, $Q = Q(x,t)$.

Assuming K_0, c, ρ are constant (uniform rod) and $Q = 0$ (no heat sources), we obtain

$$\boxed{\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}}$$

where $k = K_0(c\rho)^{-1}$ is called *thermal diffusivity*.

Initial and boundary conditions

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad x_1 \le x \le x_2.$$

Initial condition: $u(x,0) = f(x),\ \ x_1 \le x \le x_2.$

Examples of boundary conditions:

- $u(x_1,t) = u(x_2,t) = 0.$

(constant temperature at the ends)

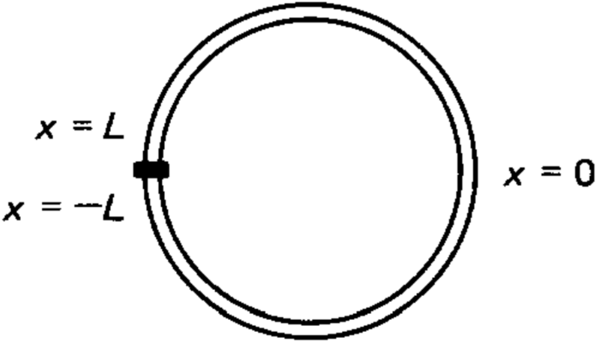
- $\frac{\partial u}{\partial x}(x_1,t) = \frac{\partial u}{\partial x}(x_2,t) = 0.$

(insulated ends)

- $u(x_1,t) = u(x_2,t),\ \ \frac{\partial u}{\partial x}(x_1,t) = \frac{\partial u}{\partial x}(x_2,t).$

(periodic boundary conditions)

Heat conduction in a thin circular ring



Separation of variables

The method applies to certain linear PDEs, for example, heat equation, wave equation, Laplace's equation.

Basic idea: to find a solution of the PDE (function of many variables) as the product of several functions, each depending only on one variable.

For example, $u(x, t) = B(x)C(t)$.

Heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$$

Suppose $u(x,t) = \phi(x)g(t)$ is a solution. Then

$$\tfrac{d^2\phi}{dx^2} = -\lambda\phi,$$

$$\tfrac{dg}{dt} = -\lambda k g,$$

where λ is a **separation constant**.

Conversely, if ϕ and g are solutions of the above ODEs for the same value of λ, then $u(x,t) = \phi(x)g(t)$ is a solution of the heat equation.

Boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(0,t) = u(L,t) = 0.$$

We are looking for solutions $u(x,t) = \phi(x)g(t)$.

PDE holds if

$$\tfrac{d^2\phi}{dx^2} = -\lambda\phi,$$
$$\tfrac{dg}{dt} = -\lambda k g$$

for the same constant λ.

Boundary conditions hold if

$$\phi(0) = \phi(L) = 0.$$

Boundary value problem:

$$\frac{d^2\phi}{dx^2} = -\lambda\phi, \qquad 0 \le x \le L,$$
$$\phi(0) = \phi(L) = 0.$$

We are looking for a nonzero solution.

This is an **eigenvalue problem**, $L(\phi) = \lambda\phi$, for a linear operator $L : V \to W$, where $L = -\frac{d^2}{dx^2}$,
$V = \{\phi \in C^2[0, L] : \phi(0) = \phi(L) = 0\}$,
$W = C[0, L]$.

The eigenvalue problem is to find all eigenvalues (and associated eigenfunctions).

Eigenvalue problem

$$\phi'' = -\lambda\phi, \quad \phi(0) = \phi(L) = 0.$$

We are looking only for real eigenvalues.

Three cases: $\lambda > 0$, $\lambda = 0$, $\lambda < 0$.

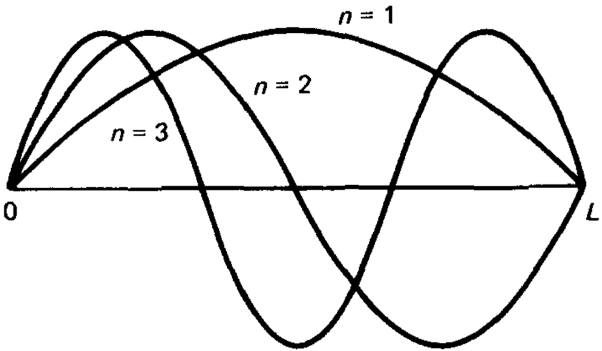
Case 1: $\lambda > 0$. $\quad \phi(x) = C_1 \cos \mu x + C_2 \sin \mu x$,
where $\lambda = \mu^2$, $\mu > 0$.

$\phi(0) = \phi(L) = 0 \implies C_1 = 0, \; C_2 \sin \mu L = 0.$

A nonzero solution exists if $\mu L = n\pi$, $n \in \mathbb{Z}$.

So $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \dots$ are eigenvalues and $\phi_n(x) = \sin \frac{n\pi x}{L}$ are corresponding eigenfunctions.

Eigenfunctions



$$\phi_n(x) = \sin \frac{n\pi x}{L}$$

Are there other eigenfunctions?

Case 2: $\lambda = 0$. $\quad \phi(x) = C_1 + C_2 x$.

$\phi(0) = \phi(L) = 0 \implies C_1 = C_1 + C_2 L = 0$

$\implies C_1 = C_2 = 0$.

Case 3: $\lambda < 0$. $\quad \phi(x) = C_1 e^{\mu x} + C_2 e^{-\mu x}$,

where $\lambda = -\mu^2$, $\mu > 0$.

$$\boxed{\cosh z = \frac{e^z + e^{-z}}{2}} \qquad \boxed{\sinh z = \frac{e^z - e^{-z}}{2}}$$

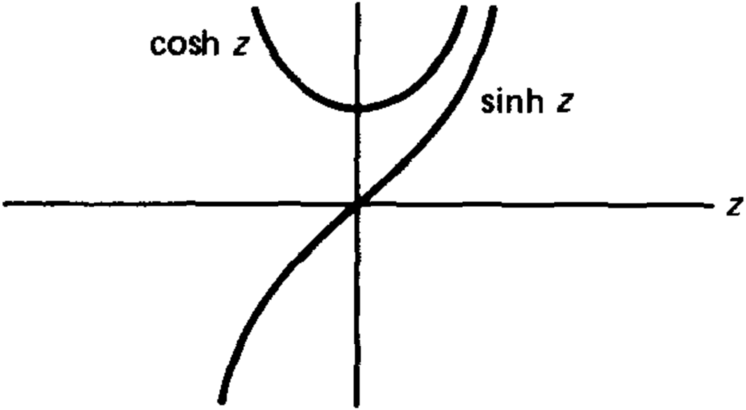
$$e^z = \cosh z + \sinh z, \qquad e^{-z} = \cosh z - \sinh z.$$

$\phi(x) = D_1 \cosh \mu x + D_2 \sinh \mu x, \qquad D_1, D_2 = \text{const.}$

$\phi(0) = 0 \implies D_1 = 0$

$\phi(L) = 0 \implies D_2 \sinh \mu L = 0 \implies D_2 = 0$

Hyperbolic functions



Summary

Eigenvalue problem: $\phi'' = -\lambda\phi$, $\phi(0) = \phi(L) = 0$.

Eigenvalues: $\lambda_n = (\frac{n\pi}{L})^2$, $n = 1, 2, \ldots$

Eigenfunctions: $\phi_n(x) = \sin\frac{n\pi x}{L}$.

Solution of the heat equation: $u(x,t) = \phi(x)g(t)$.

$$\frac{dg}{dt} = -\lambda k g \implies g(t) = C_0 \exp(-\lambda k t)$$

Theorem For $n = 1, 2, \ldots$, the function

$$\boxed{u(x,t) = e^{-\lambda_n k t}\phi_n(x) = \exp(-\frac{n^2\pi^2}{L^2}kt)\sin\frac{n\pi x}{L}}$$

is a solution of the following boundary value problem for the heat equation:

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(0,t) = u(L,t) = 0.$$

Initial-boundary value problem

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

Function $u(x,t) = e^{-\lambda_n kt}\phi_n(x)$ is a solution of the boundary value problem. Initial condition is satisfied if $f = \phi_n$. For any $B_1, B_2, \dots, B_N \in \mathbb{R}$ the function

$$u(x,t) = \sum\nolimits_{n=1}^{N} B_n e^{-\lambda_n kt}\phi_n(x)$$

is also a solution of the boundary value problem. This time the initial condition is satisfied if

$$f(x) = \sum\nolimits_{n=1}^{N} B_n\phi_n(x) = \sum\nolimits_{n=1}^{N} B_n \sin\frac{n\pi x}{L}.$$

From finite sums to series

Conjecture For suitably chosen coefficients $B_1, B_2, B_3, \dots$ the function

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n k t} \phi_n(x)$$

is a solution of the boundary value problem. This solution satisfies the initial condition with

$$f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}.$$

Theorem If $\sum_{n=1}^{\infty} |B_n| < \infty$ then the conjecture is true. Namely, $u(x,t)$ is smooth for $t > 0$ and solves the boundary value problem. Also, $u(x,t)$ is continuous for $t \geq 0$ and satisfies the initial condition.

How do we solve the initial-boundary value problem?

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad 0 \le x \le L,$$
$$u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

- Expand the function f into a series

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L}.$$

- Write the solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \exp\left(-\frac{n^2\pi^2}{L^2}kt\right) \sin\frac{n\pi x}{L}.$$

J. Fourier, The Analytical Theory of Heat
(written in 1807, published in 1822)

Orthogonal sets

Suppose V is an inner product space and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is an orthogonal set in V. For any $\mathbf{x} \in V$ let

$$\mathbf{p} = \frac{\langle \mathbf{x}, \mathbf{v}_1 \rangle}{\langle \mathbf{v}_1, \mathbf{v}_1 \rangle}\mathbf{v}_1 + \frac{\langle \mathbf{x}, \mathbf{v}_2 \rangle}{\langle \mathbf{v}_2, \mathbf{v}_2 \rangle}\mathbf{v}_2 + \cdots + \frac{\langle \mathbf{x}, \mathbf{v}_n \rangle}{\langle \mathbf{v}_n, \mathbf{v}_n \rangle}\mathbf{v}_n.$$

Then $\mathbf{p}$ is the orthogonal projection of $\mathbf{x}$ onto $\mathrm{Span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. Also, $\mathbf{p}$ is the best approximation of $\mathbf{x}$ by linear combinations $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$ relative to the distance

$$\mathrm{dist}(\mathbf{x}, \mathbf{y}) = \|\mathbf{y} - \mathbf{x}\| = \sqrt{\langle \mathbf{y} - \mathbf{x}, \mathbf{y} - \mathbf{x} \rangle}.$$

$$V = C[-\pi, \pi], \;\; \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x)\, dx.$$

$$f_1(x) = \sin x, \;\; f_2(x) = \sin 2x, \; \ldots, \;\; f_n(x) = \sin nx, \; \ldots$$

$$\langle f_n, f_m \rangle = \int_{-\pi}^{\pi} \sin(nx) \sin(mx) = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases}$$

$f_1, f_2, \ldots$ is an orthogonal set.

For any function $F \in C[-\pi, \pi]$ consider a series

$$\frac{\langle F, f_1 \rangle}{\langle f_1, f_1 \rangle} f_1(x) + \frac{\langle F, f_2 \rangle}{\langle f_2, f_2 \rangle} f_2(x) + \frac{\langle F, f_3 \rangle}{\langle f_3, f_3 \rangle} f_3(x) + \cdots$$

$$\frac{\langle F, f_1\rangle}{\langle f_1, f_1\rangle} f_1(x) + \frac{\langle F, f_2\rangle}{\langle f_2, f_2\rangle} f_2(x) + \frac{\langle F, f_3\rangle}{\langle f_3, f_3\rangle} f_3(x) + \cdots$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \dots,$$

where $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(y) \sin(ny)\, dy$.

Theorem The above series converges to some function $G \in C[-\pi, \pi]$ with respect to the distance

$$\mathrm{dist}(f, g) = \|f - g\| = \left(\int_{-\pi}^{\pi} |f(x) - g(x)|^2\, dx \right)^{1/2}.$$

Since $\sin(-nx) = -\sin(nx)$, it follows that $G(-x) = -G(x)$.

Example. $F(x) = e^x$.

In this case, the series converges to the function $G(x) = \sinh x$. Note that $G(x) = \frac{1}{2}\big(F(x) + F(-x)\big)$.

$$h_1(x) = \cos x,\ \ h_2(x) = \cos 2x,\ \dots,\ \ h_n(x) = \cos nx,\ \dots$$

$$\langle h_n, h_m \rangle = \int_{-\pi}^{\pi} \cos(nx)\cos(mx) = \begin{cases} 0, & n \neq m, \\ \pi, & n = m. \end{cases}$$

$h_1, h_2, \dots$ is an orthogonal set.

For any function $F \in C[-\pi, \pi]$ consider a series

$$\frac{\langle F, h_1 \rangle}{\langle h_1, h_1 \rangle} h_1(x) + \frac{\langle F, h_2 \rangle}{\langle h_2, h_2 \rangle} h_2(x) + \frac{\langle F, h_3 \rangle}{\langle h_3, h_3 \rangle} h_3(x) + \cdots$$

$$\frac{\langle F, h_1\rangle}{\langle h_1, h_1\rangle} h_1(x) + \frac{\langle F, h_2\rangle}{\langle h_2, h_2\rangle} h_2(x) + \frac{\langle F, h_3\rangle}{\langle h_3, h_3\rangle} h_3(x) + \cdots$$

$$= a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \ldots,$$

where $a_n = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} F(y) \cos(ny)\, dy.$

Theorem The above series converges to some function $H \in C[-\pi, \pi]$ with respect to the distance $\mathrm{dist}(f, g) = \|f - g\|$.

Since $\cos(-nx) = \cos(nx)$, it follows that $H(-x) = H(x)$. Since $\int_{-\pi}^{\pi} \cos(nx)\, dx = 0$, it follows that $\int_{-\pi}^{\pi} H(x)\, dx = 0$.

Example. $F(x) = e^x$.

In this case, the series converges to the function $H(x) = \cosh x - \pi^{-1} \sinh \pi$.

$$h_0(x) = 1,\ h_1(x) = \cos x,\ \dots,\ h_n(x) = \cos nx,\ \dots,$$
$$f_1(x) = \sin x,\ f_2(x) = \sin 2x,\ \dots,\ f_n(x) = \sin nx,\ \dots$$

This is an orthogonal set: $\langle h_0, h_0 \rangle = 2\pi$, $\langle h_n, h_n \rangle = \langle f_n, f_n \rangle = \pi$ for $n \geq 1$, while the other inner products are equal to 0.

This orthogonal set is **maximal**.

Fourier series

Definition. **Fourier series** is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

To each integrable function $F : [-\pi, \pi] \to \mathbb{R}$ we associate a Fourier series such that

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos nx\, dx,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin nx\, dx.$$

Convergence theorems

Theorem 1 Fourier series of a continuous function on $[-\pi, \pi]$ converges to this function with respect to the distance

$$\mathrm{dist}(f, g) = \|f - g\| = \left(\int_{-\pi}^{\pi} |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

However convergence in the sense of Theorem 1 need not imply pointwise convergence.

Theorem 2 Fourier series of a smooth function on $[-\pi, \pi]$ converges pointwise to this function on the open interval $(-\pi, \pi)$.

Example. Fourier series of the function $F(x) = x$.

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x\,dx = 0, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)\,dx = 0.$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)\,dx = -\frac{1}{n\pi}\int_{-\pi}^{\pi} x(\cos nx)'\,dx$$

$$= -\frac{1}{n\pi}\,x\cos(nx)\,\Big|_{-\pi}^{\pi} + \frac{1}{n\pi}\int_{-\pi}^{\pi}\cos nx\,dx$$

$$= -\frac{1}{n\pi}\cdot 2\pi\cos(n\pi) = (-1)^{n+1}\,\frac{2}{n}.$$

Example. Fourier series of the function $F(x) = x$.

$$\boxed{2 \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n}}$$

$$= 2 \left(\sin x - \frac{1}{2} \sin 2x + \frac{1}{3} \sin 3x - \frac{1}{4} \sin 4x + \cdots \right)$$

The series converges to the function $F(x)$ for any $-\pi < x < \pi$.

For $x = \pi/2$ we obtain:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$