MATH 311

Topics in Applied Mathematics

Lecture 23:
Fourier series (continued).

Fourier series

Standard **Fourier series** is a series of the form

$$a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Each term of the series is a 2π-periodic function. If the series converges, then the sum is also 2π-periodic.

More general Fourier series:

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}.$$

Each term of this series is a $2L$-periodic function.

Fourier series

To each integrable function $F : [-L, L] \to \mathbb{R}$ we associate a Fourier series

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

such that

$$a_0 = \frac{1}{2L} \int_{-L}^{L} F(x)\, dx$$

and for $n \geq 1$,

$$a_n = \frac{1}{L} \int_{-L}^{L} F(x) \cos \frac{n\pi x}{L}\, dx,$$

$$b_n = \frac{1}{L} \int_{-L}^{L} F(x) \sin \frac{n\pi x}{L}\, dx.$$

Example. Fourier series of the function $F(x) = x$ on the interval $[-\pi, \pi]$ is

$$2 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$$

$$= 2\left(\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \cdots \right).$$

Fourier series of the same function $F(x) = x$ on an interval $[-L, L]$ is

$$\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{L}.$$

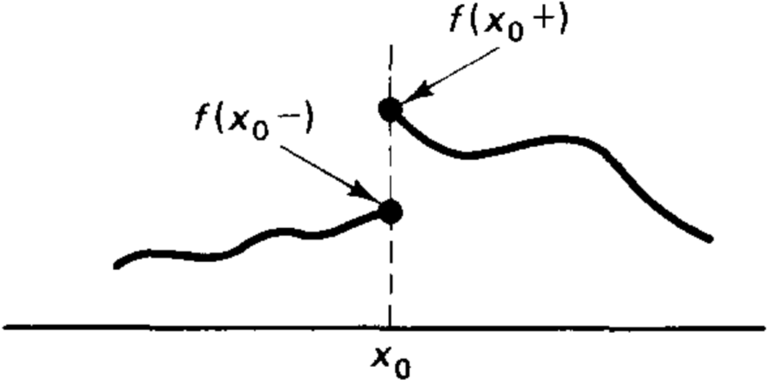
Convergence theorems

Theorem 1 Fourier series of a continuous function on $[-L, L]$ converges to this function with respect to the distance

$$\mathrm{dist}(f, g) = \|f - g\| = \left(\int_{-L}^{L} |f(x) - g(x)|^2 \, dx \right)^{1/2}.$$

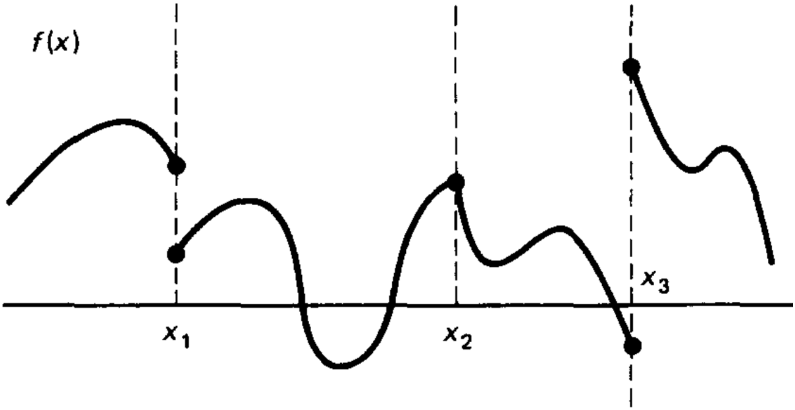
However convergence in the sense of Theorem 1 need not imply pointwise convergence.

Theorem 2 Fourier series of a smooth function on $[-L, L]$ converges pointwise to this function on the open interval $(-L, L)$.

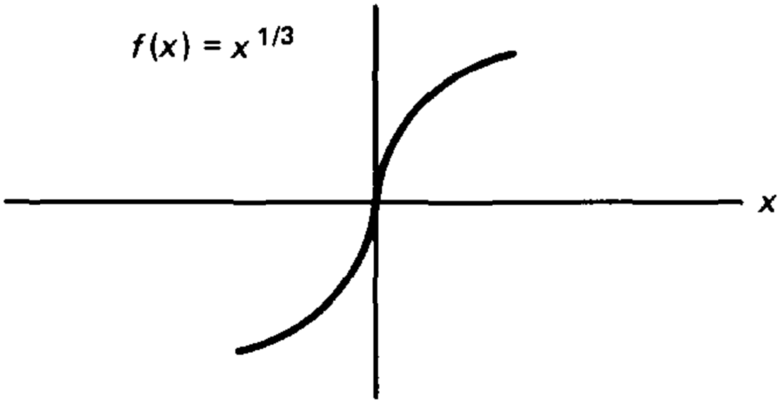


Jump discontinuity

Piecewise continuous = finitely many jump discontinuities



Piecewise smooth function
(both function and its derivative
are piecewise continuous)



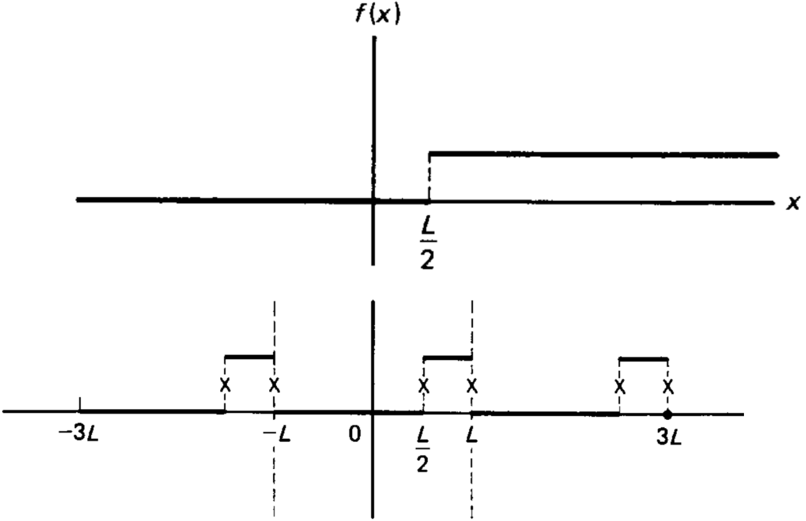
Continuous, but not piecewise smooth function

Convergence theorem

Suppose $f : [-L, L] \to \mathbb{R}$ is a piecewise smooth function. Let $F : \mathbb{R} \to \mathbb{R}$ be the $2L$-**periodic extension** of f. That is, F is $2L$-periodic and $F(x) = f(x)$ for $-L < x \leq L$. Clearly, F is also piecewise smooth.

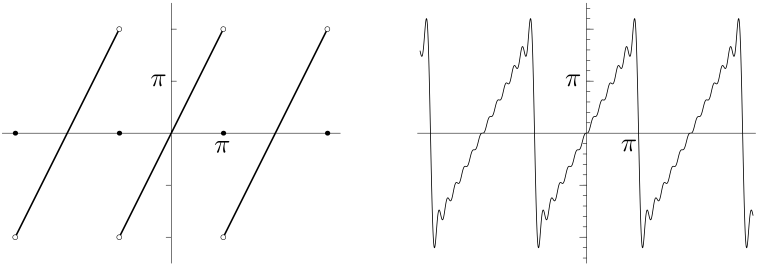
Theorem The Fourier series of the function f converges everywhere. The sum at a point x is equal to $F(x)$ if F is continuous at x. Otherwise the sum is equal to

$$\frac{F(x-) + F(x+)}{2}.$$



Function and its Fourier series

Gibbs' phenomenon



Left graph: Fourier series of $F(x) = 2x$.
Right graph: 12th partial sum of the series.

The maximal value of the nth partial sum for large n is about 17.9% higher than the maximal value of the series. This is the so-called **Gibbs' overshoot**.

Fourier sine and cosine series

Suppose $f(x)$ is an integrable function on $[0, L]$.
The Fourier sine series of f

$$\sum_{n=1}^{\infty} B_n \sin \tfrac{n\pi x}{L}$$

and the Fourier cosine series of f

$$A_0 + \sum_{n=1}^{\infty} A_n \cos \tfrac{n\pi x}{L}$$

are defined as follows:

$$B_n = \tfrac{2}{L} \int_0^L f(x) \sin \tfrac{n\pi x}{L} \, dx;$$

$$A_0 = \tfrac{1}{L} \int_0^L f(x) \, dx, \quad A_n = \tfrac{2}{L} \int_0^L f(x) \cos \tfrac{n\pi x}{L} \, dx, \ \ n \geq 1.$$

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} a_n \cos \tfrac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin \tfrac{n\pi x}{L},$$

where

$$a_0 = \tfrac{1}{2L} \int_{-L}^{L} f(x)\, dx, \quad a_n = \tfrac{1}{L} \int_{-L}^{L} f(x) \cos \tfrac{n\pi x}{L}\, dx, \quad n \geq 1,$$

$$b_n = \tfrac{1}{L} \int_{-L}^{L} f(x) \sin \tfrac{n\pi x}{L}\, dx.$$

If f is **odd**, $f(-x) = -f(x)$, then $a_n = 0$ and

$$b_n = \tfrac{2}{L} \int_{0}^{L} f(x) \sin \tfrac{n\pi x}{L}\, dx.$$

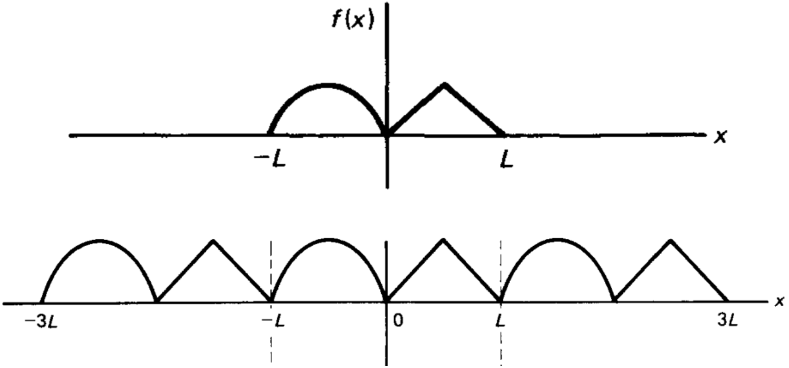
Similarly, if f is **even**, $f(-x) = f(x)$, then $b_n = 0$ and $a_n = A_n$.

Proposition (i) The Fourier series of an odd function $f : [-L, L] \to \mathbb{R}$ coincides with its Fourier sine series on $[0, L]$.

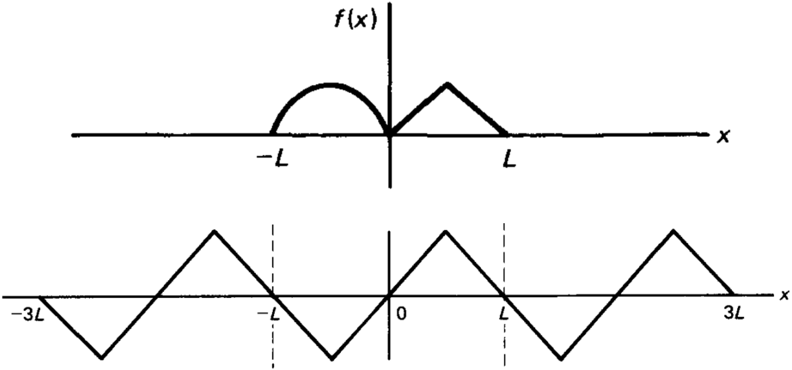
(ii) The Fourier series of an even function $f : [-L, L] \to \mathbb{R}$ coincides with its Fourier cosine series on $[0, L]$.

Conversely, the Fourier sine series of a function $f : [0, L] \to \mathbb{R}$ is the Fourier series of its **odd extension** to $[-L, L]$.

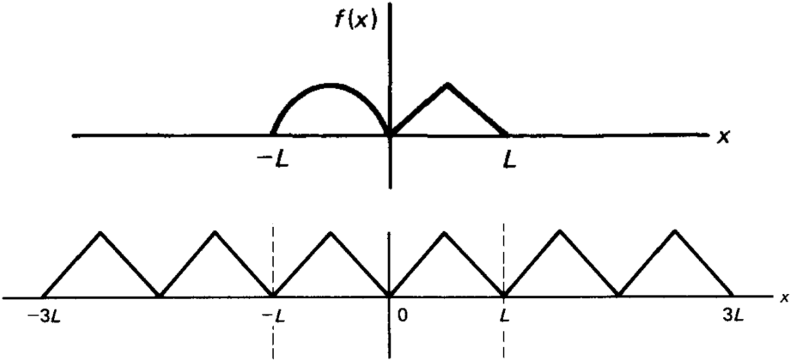
The Fourier cosine series of f is the Fourier series of its **even extension** to $[-L, L]$.



Fourier series
(2L-periodic)



Fourier sine series
(2L-periodic and odd)



Fourier cosine series
(2L-periodic and even)

Example. Fourier cosine series of $F(x) = x$.

$$A_0 = \frac{1}{\pi}\int_0^{\pi} x\,dx = \frac{\pi}{2},$$

$$A_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = \frac{2}{n\pi}\int_0^{\pi} x(\sin nx)'\,dx$$

$$= \frac{2}{n\pi}\,x\sin(nx)\,\Big|_0^{\pi} - \frac{2}{n\pi}\int_0^{\pi}\sin nx\,dx = -\frac{2}{n\pi}\int_0^{\pi}\sin nx\,dx$$

$$= \frac{2}{n^2\pi}\cos(nx)\,\Big|_0^{\pi} = \begin{cases} -4/(n^2\pi), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}$$

$$\boxed{x \sim \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \frac{\cos 7x}{7^2} + \cdots\right)}$$

Example. Fourier series of the function $f(x) = x^2$.

Proposition Fourier series of an odd function contains only sines, while Fourier series of an even function contains only cosines and a constant term.

Theorem Suppose that a function $f : [-\pi, \pi] \to \mathbb{R}$ is continuous, piecewise smooth, and $f(-\pi) = f(\pi)$.

Then the Fourier series of f' can be obtained via **term-by-term differentiation** of the Fourier series of f.

Example. Fourier series of the function $f(x) = x^2$.

$$x^2 \sim a_0 + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + \cdots$$

Term-by-term differentiation yields

$$-a_1 \sin x - 2a_2 \sin 2x - 3a_3 \sin 3x - 4a_4 \sin 4x - \cdots$$

This should be the Fourier series of $f'(x) = 2x$, which is

$$2x \sim 4\left(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \tfrac{1}{4}\sin 4x + \cdots\right).$$

Hence $a_n = (-1)^n \frac{4}{n^2}$ for $n \geq 1$.

It remains to find $a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{\pi^2}{3}$.

Example. Fourier series of the function $f(x) = x^2$.

$$x^2 \sim \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

$$= \frac{\pi^2}{3} + 4\left(-\cos x + \frac{1}{4}\cos 2x - \frac{1}{9}\cos 3x + \frac{1}{16}\cos 4x - \cdots\right)$$

The series converges to $f(x)$ for any $-\pi \le x \le \pi$.

For $x = 0$ we obtain: $\dfrac{\pi^2}{12} = 1 - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \cdots$

For $x = \pi$ we obtain: $\dfrac{\pi^2}{6} = 1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$

Hilbert basis

Let V be an infinite-dimensional inner product space. Suppose that $f_1, f_2, \dots$ is a **maximal orthogonal set** in V, i.e., there is no nonzero vector $f \in V$ such that $\langle f, f_n \rangle = 0$, $n = 1, 2, \dots$.

Then $f_1, f_2, \dots$ is a **Hilbert basis** for V, which means that any $g \in V$ can be expanded into a series

$$g = \sum_{n=1}^{\infty} c_n f_n \quad (c_n \in \mathbb{R})$$

that converges with respect to the distance $\mathrm{dist}(f, g) = \|f - g\| = \sqrt{\langle f - g, f - g \rangle}$.

$$g = \sum_{n=1}^{\infty} c_n f_n \quad \Longrightarrow \quad \langle g, h \rangle = \sum_{n=1}^{\infty} c_n \langle f_n, h \rangle, \ \ h \in V.$$

In particular, $\langle g, f_m \rangle = \sum_{n=1}^{\infty} c_n \langle f_n, f_m \rangle = c_m \langle f_m, f_m \rangle.$

$\Longrightarrow$ the expansion is unique: $c_m = \dfrac{\langle g, f_m \rangle}{\langle f_m, f_m \rangle}.$

Also,

$$\langle g, g \rangle = \sum_{n=1}^{\infty} c_n \langle f_n, g \rangle = \sum_{n=1}^{\infty} |c_n|^2 \langle f_n, f_n \rangle.$$

$$\boxed{\langle g, g \rangle = \sum_{n=1}^{\infty} \frac{|\langle g, f_n \rangle|^2}{\langle f_n, f_n \rangle}}$$

(Parseval's equality)

$$V = C[a,b],\ \ \langle f,g\rangle = \int_a^b f(x)g(x)\,dx.$$

$$h_0(x) = 1,\ h_1(x) = \cos\tfrac{\pi x}{L},\ \ldots,\ h_n(x) = \cos\tfrac{n\pi x}{L},\ \ldots,$$

$$f_1(x) = \sin\tfrac{\pi x}{L},\ f_2(x) = \sin\tfrac{2\pi x}{L},\ \ldots,\ f_n(x) = \sin\tfrac{n\pi x}{L}, \ldots$$

Functions h_n $(n \geq 0)$ and f_n $(n \geq 1)$ form a maximal orthogonal set in $C[-L,L]$. Functions h_n $(n \geq 0)$ form a maximal orthogonal set in $C[0,L]$. Functions f_n $(n \geq 1)$ form another maximal orthogonal set in $C[0,L]$.

Parseval's equality for Fourier sine series:

$$\boxed{\frac{2}{L}\int_0^L |f(x)|^2\,dx = \sum\nolimits_{n=1}^{\infty} |c_n|^2,}$$

where $f(x) \sim \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L}$.

Example. $f(x) = x, \;\; 0 \le x \le \pi.$

$$f(x) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx$$

Parseval's equality:

$$\frac{2}{\pi} \int_0^{\pi} x^2 \, dx = \sum_{n=1}^{\infty} |c_n|^2 = \sum_{n=1}^{\infty} \frac{4}{n^2}.$$

$$\frac{2}{\pi} \cdot \frac{\pi^3}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$