MATH 311

Topics in Applied Mathematics

Lecture 24:
Heat equation (continued).
Bessel functions.

System of linear ODEs

$$\begin{cases} \frac{dx}{dt} = 2x + y, \\ \frac{dy}{dt} = x + 2y, \end{cases} \qquad x(0) = y(0) = 1.$$

This initial value problem can be rewritten in vector form:

$$\frac{d\mathbf{v}}{dt} = \mathcal{L}(\mathbf{v}), \qquad \mathbf{v}(0) = \mathbf{v}_0,$$

where $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$, $\mathcal{L}(\mathbf{v}) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{v}$, $\mathbf{v}_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

If $\mathcal{L}(\mathbf{w}) = \lambda \mathbf{w}$, then the system has a solution $\mathbf{v}(t) = e^{\lambda t}\mathbf{w}$.

$$\frac{d\mathbf{v}}{dt} = \mathcal{L}(\mathbf{v}), \quad \mathbf{v}(0) = \mathbf{v}_0.$$

Suppose that the linear operator $\mathcal{L}$ is diagonalizable, i.e., there exists a basis $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ formed by its eigenvectors: $\mathcal{L}(\mathbf{w}_i) = \lambda_i \mathbf{w}_i$.

Then the initial value problem is solved as follows:

- expand the initial value $\mathbf{v}_0$ into a linear combination
$$\mathbf{v}_0 = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \cdots + c_n\mathbf{w}_n;$$
- write down the solution:
$$\mathbf{v}(t) = c_1 e^{\lambda_1 t}\mathbf{w}_1 + c_2 e^{\lambda_2 t}\mathbf{w}_2 + \cdots + c_n e^{\lambda_n t}\mathbf{w}_n.$$

If, in addition, $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_n$ is an orthogonal set, then
$$c_i = \frac{\mathbf{v}_0 \cdot \mathbf{w}_i}{\mathbf{w}_i \cdot \mathbf{w}_i}, \quad i = 1, 2, \dots, n.$$

Heat equation

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x) \quad (0 \le x \le L),$$
$$u(0,t) = u(L,t) = 0.$$

Consider vector spaces $V = \{\phi \in C^2[0,L] : \phi(0) = \phi(L) = 0\}$, $W = C[0,L]$ and a linear operator $\mathcal{L} : V \to W$ given by $\mathcal{L}(\phi) = k\phi''$. Then the initial-boundary value problem for the heat equation can be represented as the initial value problem for a linear ODE on the space V:

$$\frac{dF}{dt} = \mathcal{L}(F), \quad F(0) = f.$$

The space V is endowed with an inner product

$$\langle f, g \rangle = \int_0^L f(x)g(x)\,dx.$$

$$\mathcal{L}(\phi) = k\phi'', \;\; \phi \in V = \{\phi \in C^2[0,L] : \phi(0)=\phi(L)=0\}.$$

Eigenvalues of $\mathcal{L}$: $\lambda_n = -k(\frac{n\pi}{L})^2$, $n = 1, 2, \dots$

Eigenfunctions: $\phi_n(x) = \sin \frac{n\pi x}{L}$.

The eigenfunctions $\phi_1, \phi_2, \dots$ form a maximal orthogonal set (Hilbert basis) in the space V.

To solve the initial-boundary value problem for the heat equation,

- expand the initial data f into a series

$$f(x) = \sum_{n=1}^{\infty} B_n \phi_n(x), \qquad B_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}$$

(this is the Fourier sine series of f on $[0, L]$);

- write down the solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{\lambda_n t} \phi_n(x) = \sum_{n=1}^{\infty} B_n \exp\Bigl(-\frac{n^2\pi^2}{L^2}kt\Bigr) \sin \frac{n\pi x}{L}.$$

Heat equation: insulated ends

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x) \quad (0 \le x \le L),$$

$$\tfrac{\partial u}{\partial x}(0,t) = \tfrac{\partial u}{\partial x}(L,t) = 0.$$

- Expand f into the Fourier cosine series on $[0, L]$:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L}.$$

- Write down the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\Bigl(-\frac{n^2\pi^2}{L^2}kt\Bigr) \cos\frac{n\pi x}{L}.$$

Heat equation: circular ring

$$\frac{\partial u}{\partial t} = k\,\frac{\partial^2 u}{\partial x^2}, \qquad u(x,0) = f(x) \qquad (-L \le x \le L),$$

$$u(-L,t) = u(L,t), \qquad \tfrac{\partial u}{\partial x}(-L,t) = \tfrac{\partial u}{\partial x}(L,t).$$

- Expand f into the Fourier series on $[-L, L]$:

$$f(x) = A_0 + \sum_{n=1}^{\infty}\Big(A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}\Big).$$

- Write down the solution:

$$u(x,t) = A_0 + \sum_{n=1}^{\infty} \exp\!\big(-\tfrac{n^2\pi^2}{L^2}kt\big)\big(A_n \cos\tfrac{n\pi x}{L} + B_n \sin\tfrac{n\pi x}{L}\big).$$

2-dimensional heat equation

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right), \quad (x, y) \in D,$$

$$u(x, y, 0) = f(x, y), \quad (x, y) \in D,$$

Boundary condition: $u|_{\partial D} = 0$,

i.e., $u(x, y, t) = 0$ for $(x, y) \in \partial D$.

(Dirichlet condition)

Alternative boundary condition: $\displaystyle \frac{\partial u}{\partial n}\Big|_{\partial D} = 0$,

where $\frac{\partial u}{\partial n} = \nabla u \cdot \mathbf{n}$ is the **normal derivative**.

(Neumann condition)

First we need to solve an eigenvalue problem:

$$\nabla^2 \phi = -\lambda \phi, \quad \phi|_{\partial D} = 0.$$

(Dirichlet Laplacian)

There exist infinitely many eigenvalues $0 < \lambda_1 \leq \lambda_2 \leq \dots$ The associated eigenfunctions $\phi_1(x, y), \phi_2(x, y), \dots$ can be chosen so that they are orthogonal relative to the inner product

$$\langle f, g \rangle = \iint_D f(x, y) g(x, y)\, dx\, dy.$$

To solve the initial-boundary value problem for the heat equation, expand the initial data f in the eigenfunctions:

$$f = \sum_{n=1}^{\infty} c_n \phi_n, \qquad c_n = \frac{\langle f, \phi_n \rangle}{\langle \phi_n, \phi_n \rangle}.$$

Then

$$u(x, y, t) = \sum_{n=1}^{\infty} c_n e^{-\lambda_n k t} \phi_n(x, y).$$

Example.

$$\nabla^2\phi = -\lambda\phi \ \text{ in } \ D = \{(x,y) \mid 0 < x < L,\ 0 < y < H\},$$

$$\phi(0,y) = \phi(L,y) = 0, \quad \phi(x,0) = \phi(x,H) = 0.$$

This problem can be solved by separation of variables.

Eigenfunctions $\phi_{nm}(x,y) = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{H}$, $n, m \geq 1$.

Corresponding eigenvalues: $\lambda_{nm} = (\frac{n\pi}{L})^2 + (\frac{m\pi}{H})^2$.

The expansion in eigenfunctions of the Dirichlet Laplacian in a rectangle is called the **double Fourier sine series**.

Eigenvalues of the Laplacian in a circle

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \;\text{ in }\; D = \{(x,y) : x^2 + y^2 \le R^2\},$$

$$\phi|_{\partial D} = 0.$$

In polar coordinates (r, θ):

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \lambda\phi = 0$$

$$(0 < r < R,\ -\pi < \theta < \pi),$$

$$\phi(R,\theta) = 0 \quad (-\pi < \theta < \pi).$$

Additional boundary conditions:

$$|\phi(0,\theta)| < \infty \quad (-\pi < \theta < \pi),$$

$$\phi(r,-\pi) = \phi(r,\pi), \;\; \tfrac{\partial\phi}{\partial\theta}(r,-\pi) = \tfrac{\partial\phi}{\partial\theta}(r,\pi) \quad (0 < r < R).$$

Separation of variables: $\phi(r,\theta) = f(r)h(\theta)$.
Substitute this into the equation:

$$f''(r)h(\theta) + r^{-1}f'(r)h(\theta) + r^{-2}f(r)h''(\theta) + \lambda f(r)h(\theta) = 0.$$

Divide by $f(r)h(\theta)$ and multiply by r^2:

$$\frac{r^2f''(r) + r\,f'(r) + \lambda r^2 f(r)}{f(r)} + \frac{h''(\theta)}{h(\theta)} = 0.$$

It follows that

$$\frac{r^2 f''(r) + r\, f'(r) + \lambda r^2 f(r)}{f(r)} = -\frac{h''(\theta)}{h(\theta)} = \mu = \text{const.}$$

The variables have been separated:

$$r^2 f'' + r f' + (\lambda r^2 - \mu) f = 0,$$
$$h'' = -\mu h.$$

Boundary conditions $\phi(R,\theta) = 0$ and $|\phi(0,\theta)| < \infty$ hold if $f(R) = 0$ and $|f(0)| < \infty$.

Boundary conditions $\phi(r,-\pi) = \phi(r,\pi)$ and $\frac{\partial\phi}{\partial\theta}(r,-\pi) = \frac{\partial\phi}{\partial\theta}(r,\pi)$ hold if $h(-\pi) = h(\pi)$ and $h'(-\pi) = h'(\pi)$.

Eigenvalue problem:

$$h'' = -\mu h, \quad h(-\pi) = h(\pi),\ h'(-\pi) = h'(\pi).$$

Eigenvalues: $\mu_m = m^2$, $m = 0, 1, 2, \ldots$.
$\mu_0 = 0$ is simple, the others are of multiplicity 2.

Eigenfunctions: $h_0 = 1$, $h_m(\theta) = \cos m\theta$ and $\tilde{h}_m(\theta) = \sin m\theta$ for $m \geq 1$.

Dependence on r:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \qquad f(R) = 0,\ |f(0)| < \infty.$$

We may assume that $\mu = m^2$, $m = 0, 1, 2, \dots$.
Also, we are only interested in the case $\lambda > 0$.

New variable $z = \sqrt{\lambda} \cdot r$ removes dependence on λ:

$$\frac{df}{dr} = \sqrt{\lambda}\,\frac{df}{dz}, \qquad \frac{d^2 f}{dr^2} = \lambda\,\frac{d^2 f}{dz^2}.$$

$$\boxed{z^2\,\frac{d^2 f}{dz^2} + z\,\frac{df}{dz} + (z^2 - m^2)f = 0}$$

This is **Bessel's differential equation** of order m.
Solutions are called **Bessel functions** of order m.

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0$$

Solutions are well behaved in the interval $(0, \infty)$.

Let f_1 and f_2 be linearly independent solutions. Then the general solution is $f = c_1 f_1 + c_2 f_2$, where c_1, c_2 are constants.

We need to determine the behavior of solutions as $z \to 0$ and as $z \to \infty$.

In a neighborhood of 0, Bessel's equation is a small perturbation of the equidimensional equation

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - m^2 f = 0.$$

Equidimensional equation:

$$z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} - m^2 f = 0.$$

For $m > 0$, the general solution is
$f(z) = c_1 z^m + c_2 z^{-m}$, where c_1, c_2 are constants.

For $m = 0$, the general solution is
$f(z) = c_1 + c_2 \log z$, where c_1, c_2 are constants.

We hope that Bessel functions are close to solutions of the equidimensional equation as $z \to 0$.

Theorem For any $m > 0$ there exist Bessel functions f_1 and f_2 of order m such that

$$f_1(z) \sim z^m \quad \text{and} \quad f_2(z) \sim z^{-m} \quad \text{as} \quad z \to 0.$$

Also, there exist Bessel functions f_1 and f_2 of order 0 such that

$$f_1(z) \sim 1 \quad \text{and} \quad f_2(z) \sim \log z \quad \text{as} \quad z \to 0.$$

Remarks. (i) f_1 and f_2 are linearly independent.
(ii) f_1 is determined uniquely while f_2 is not.

$J_m(z)$: **Bessel function of the first kind**,

$Y_m(z)$: **Bessel function of the second kind**.

$J_m(z)$ and $Y_m(z)$ are certain linearly independent Bessel functions of order m.

$J_m(z)$ is regular while $Y_m(z)$ has singularity at 0.

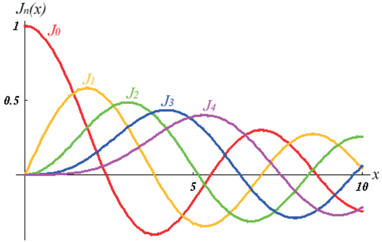
$J_m(z)$ and $Y_m(z)$ are **special functions**.

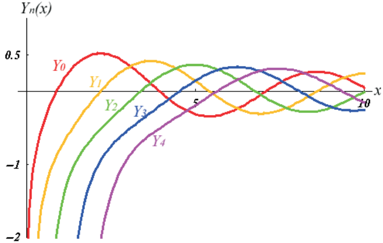
As $z \to 0$, we have for $m > 0$

$$J_m(z) \sim \frac{1}{2^m\, m!} z^m, \quad Y_m(z) \sim -\frac{2^m(m-1)!}{\pi} z^{-m}.$$

Also, $J_0(z) \sim 1, \quad Y_0(z) \sim \dfrac{2}{\pi} \log z.$

Bessel functions of the 1st and 2nd kind





$J_m(z)$ is uniquely determined by its asymptotics as $z \to 0$. Original definition by Bessel:

$$J_m(z) = \frac{1}{\pi}\int_0^{\pi} \cos(z \sin\tau - m\tau)\, d\tau.$$

Behavior of the Bessel functions as $z \to \infty$ does not depend on the order m. Any Bessel function f satisfies

$$f(z) = Az^{-1/2}\cos(z - B) + O(z^{-1}) \quad \text{as} \quad z \to \infty,$$

where A, B are constants.

The function f is uniquely determined by A, B, and its order m.