MATH 311

Topics in Applied Mathematics

Lecture 25:
Bessel functions (continued).

Bessel's differential equation of order $m \geq 0$:

$$\boxed{z^2 \frac{d^2 f}{dz^2} + z \frac{df}{dz} + (z^2 - m^2) f = 0}$$

The equation is considered on the interval $(0, \infty)$.

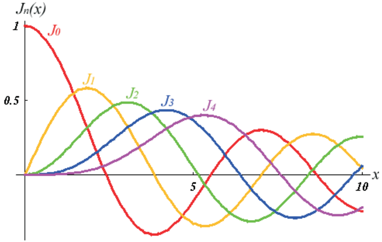
Solutions are called **Bessel functions** of order m.

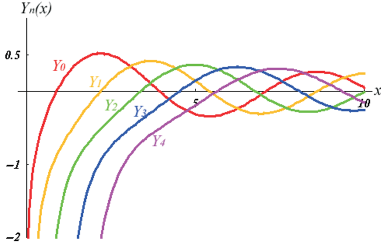
$J_m(z)$: **Bessel function of the first kind**,

$Y_m(z)$: **Bessel function of the second kind**.

The general Bessel function of order m is $f(z) = c_1 J_m(z) + c_2 Y_m(z)$, where c_1, c_2 are constants.

Bessel functions of the 1st and 2nd kind





Asymptotics at the origin

$J_m(z)$ is regular while $Y_m(z)$ has a singularity at 0.

As $z \to 0$, we have for any integer $m > 0$

$$J_m(z) \sim \frac{1}{2^m\, m!} z^m, \qquad Y_m(z) \sim -\frac{2^m (m-1)!}{\pi} z^{-m}.$$

Also, $\quad J_0(z) \sim 1, \qquad Y_0(z) \sim \dfrac{2}{\pi} \log z.$

To get the asymptotics for a noninteger m, we replace $m!$ by $\Gamma(m+1)$ and $(m-1)!$ by $\Gamma(m)$.

$J_m(z)$ is uniquely determined by this asymptotics while $Y_m(z)$ is not.

Asymptotics at infinity

As $z \to \infty$, we have

$$J_m(z) = \sqrt{\frac{2}{\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) + O(z^{-1}),$$

$$Y_m(z) = \sqrt{\frac{2}{\pi z}} \sin\left(z - \frac{\pi}{4} - \frac{m\pi}{2}\right) + O(z^{-1}).$$

Both $J_m(z)$ and $Y_m(z)$ are uniquely determined by this asymptotics.

For $m = 1/2$, these are exact formulas.

Original definition by Bessel (only for integer m):

$$J_m(z) = \frac{1}{\pi}\int_0^\pi \cos(z\sin\tau - m\tau)\,d\tau$$

$$= \frac{1}{\pi}\int_0^\pi \cos(z\sin\tau)\cos(m\tau)\,d\tau + \frac{1}{\pi}\int_0^\pi \sin(z\sin\tau)\sin(m\tau)\,d\tau.$$

The first integral is 0 for any odd m while the second integral is 0 for any even m. It follows that

$$\cos(z\sin\tau) = J_0(z) + 2\sum_{n=1}^\infty J_{2n}(z)\cos(2n\tau),$$

$$\sin(z\sin\tau) = 2\sum_{n=1}^\infty J_{2n-1}(z)\sin\big((2n-1)\tau\big).$$

Zeros of Bessel functions

Let $0 < j_{m,1} < j_{m,2} < \dots$ be zeros of $J_m(z)$ and $0 < y_{m,1} < y_{m,2} < \dots$ be zeros of $Y_m(z)$.

Let $0 \le j'_{m,1} < j'_{m,2} < \dots$ be zeros of $J'_m(z)$ and $0 < y'_{m,1} < y'_{m,2} < \dots$ be zeros of $Y'_m(z)$.

(We let $j'_{0,1} = 0$ while $j'_{m,1} > 0$ if $m > 0$.)

Then the zeros are interlaced:

$$m \le j'_{m,1} < y_{m,1} < y'_{m,1} < j_{m,1} <$$
$$< j'_{m,2} < y_{m,2} < y'_{m,2} < j_{m,2} < \dots$$

Asymptotics of the nth zeros as $n \to \infty$:

$$j'_{m,n} \approx y_{m,n} \sim (n + \tfrac{1}{2}m - \tfrac{3}{4})\pi,$$
$$y'_{m,n} \approx j_{m,n} \sim (n + \tfrac{1}{2}m - \tfrac{1}{4})\pi.$$

Dirichlet Laplacian in a circle

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \ \text{ in } \ D = \{(x,y) : x^2 + y^2 \le R^2\},$$

$$\phi|_{\partial D} = 0.$$

Separation of variables in polar coordinates: $\phi(r,\theta) = f(r)h(\theta)$. Reduces the problem to two one-dimensional eigenvalue problems:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \qquad f(R) = 0, \ |f(0)| < \infty;$$

$$h'' = -\mu h, \qquad h(-\pi) = h(\pi), \ h'(-\pi) = h'(\pi).$$

The latter problem has eigenvalues $\mu_m = m^2$, $m = 0, 1, 2, \dots$, and eigenfunctions $h_0 = 1$, $h_m(\theta) = \cos m\theta$, $\tilde{h}_m(\theta) = \sin m\theta$, $m \ge 1$.

The 1st intermediate eigenvalue problem:

$$r^2 f'' + rf' + (\lambda r^2 - m^2) f = 0, \quad f(R) = 0, \, |f(0)| < \infty.$$

New variable $z = \sqrt{\lambda} \cdot r$ reduces the equation to Bessel's equation of order m. Hence the general solution is $f(r) = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r)$, where c_1, c_2 are constants.

Singular condition $|f(0)| < \infty$ holds if $c_2 = 0$.
Nonzero solution exists if $J_m(\sqrt{\lambda}\, R) = 0$.

Thus there are infinitely many eigenvalues $\lambda_{m,1}, \lambda_{m,2}, \dots,$ where $\sqrt{\lambda_{m,n}}\, R = j_{m,n}$, i.e., $\lambda_{m,n} = (j_{m,n}/R)^2$.
Associated eigenfunctions: $f_{m,n}(r) = J_m(j_{m,n}\, r/R)$.

The eigenfunctions $f_{m,n}(r) = J_m(j_{m,n}\, r/R)$ are orthogonal relative to the inner product

$$\langle f, g\rangle_r = \int_0^R f(r)\, g(r)\, r\, dr.$$

Any piecewise continuous function g on $[0, R]$ is expanded into a **Fourier-Bessel series**

$$g(r) = \sum_{n=1}^{\infty} c_n\, J_m\Big(j_{m,n}\, \frac{r}{R}\Big), \qquad c_n = \frac{\langle g, f_{m,n}\rangle_r}{\langle f_{m,n}, f_{m,n}\rangle_r},$$

that converges in the mean (with weight r).
If g is piecewise smooth, then the series converges at its points of continuity.

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \;\text{ in }\; D = \{(x,y) : x^2 + y^2 \le R^2\},$$
$$\phi|_{\partial D} = 0.$$

Eigenvalues: $\lambda_{m,n} = (j_{m,n}/R)^2$, where $m = 0, 1, 2, \dots,\ n = 1, 2, \dots$, and $j_{m,n}$ is the nth positive zero of the Bessel function J_m.

Eigenfunctions: $\phi_{0,n}(r,\theta) = J_0(j_{0,n}\, r/R)$.

For $m \ge 1$, $\phi_{m,n}(r,\theta) = J_m(j_{m,n}\, r/R)\cos m\theta$ and $\tilde{\phi}_{m,n}(r,\theta) = J_m(j_{m,n}\, r/R)\sin m\theta$.

Neumann Laplacian in a circle

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \ \text{ in } \ D = \{(x, y) : x^2 + y^2 \le R^2\},$$
$$\frac{\partial\phi}{\partial n}\Big|_{\partial D} = 0.$$

Again, separation of variables in polar coordinates, $\phi(r, \theta) = f(r)h(\theta)$, reduces the problem to two one-dimensional eigenvalue problems:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \quad f'(R) = 0, \ |f(0)| < \infty;$$

$$h'' = -\mu h, \quad h(-\pi) = h(\pi), \ h'(-\pi) = h'(\pi).$$

The 2nd problem has eigenvalues $\mu_m = m^2$, $m = 0, 1, 2, \ldots,$ and eigenfunctions $h_0 = 1$, $h_m(\theta) = \cos m\theta$, $\tilde{h}_m(\theta) = \sin m\theta$, $m \ge 1$.

The 1st one-dimensional eigenvalue problem:

$$r^2 f'' + rf' + (\lambda r^2 - m^2)f = 0, \quad f'(R) = 0, \ |f(0)| < \infty.$$

For $\lambda > 0$, the general solution of the equation is $f(r) = c_1 J_m(\sqrt{\lambda}\, r) + c_2 Y_m(\sqrt{\lambda}\, r)$, where c_1, c_2 are constants.

Singular condition $|f(0)| < \infty$ holds if $c_2 = 0$.
Nonzero solution exists if $J_m'(\sqrt{\lambda}\, R) = 0$.

Thus there are infinitely many eigenvalues $\lambda_{m,1}, \lambda_{m,2}, \dots,$ where $\sqrt{\lambda_{m,n}}\, R = j'_{m,n}$, i.e., $\lambda_{m,n} = (j'_{m,n}/R)^2$.
Associated eigenfunctions: $f_{m,n}(r) = J_m(j'_{m,n}\, r/R)$.

$\lambda = 0$ is an eigenvalue only for $m = 0$.

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \;\text{ in }\; D = \{(x,y) : x^2 + y^2 \le R^2\},$$
$$\frac{\partial\phi}{\partial n}\Big|_{\partial D} = 0.$$

Eigenvalues: $\lambda_{m,n} = (j'_{m,n}/R)^2$, where $m = 0, 1, 2, \dots,\ n = 1, 2, \dots,$ and $j'_{m,n}$ is the nth positive zero of J'_m (exception: $j'_{0,1} = 0$).

Eigenfunctions: $\phi_{0,n}(r,\theta) = J_0(j'_{0,n}\, r/R)$.
In particular, $\phi_{0,1} = 1$.
For $m \ge 1$, $\phi_{m,n}(r,\theta) = J_m(j'_{m,n}\, r/R) \cos m\theta$ and $\tilde\phi_{m,n}(r,\theta) = J_m(j'_{m,n}\, r/R) \sin m\theta$.

Laplacian in a circular sector

Eigenvalue problem:

$\nabla^2\phi + \lambda\phi = 0$ in $D = \{(r,\theta) : r < R,\ 0 < \theta < L\}$,

$\phi|_{\partial D} = 0.$

Again, separation of variables in polar coordinates, $\phi(r,\theta) = f(r)h(\theta)$, reduces the problem to two one-dimensional eigenvalue problems:

$$r^2 f'' + rf' + (\lambda r^2 - \mu)f = 0, \qquad f(0) = f(R) = 0;$$

$$h'' = -\mu h, \qquad h(0) = h(L) = 0.$$

The 2nd problem has eigenvalues $\mu_m = (\frac{m\pi}{L})^2$, $m = 1, 2, \dots$, and eigenfunctions $h_m(\theta) = \sin\frac{m\pi\theta}{L}$.

The 1st one-dimensional eigenvalue problem:

$$r^2 f'' + r f' + (\lambda r^2 - \nu^2) f = 0, \quad f(0) = f(R) = 0.$$

Here $\nu^2 = \mu_m$. We may assume that $\lambda > 0$.
The general solution of the equation is
$f(r) = c_1 J_\nu(\sqrt{\lambda}\, r) + c_2 Y_\nu(\sqrt{\lambda}\, r)$, where c_1, c_2 are constants.

Boundary condition $f(0) = 0$ holds if $c_2 = 0$.
Nonzero solution exists if $J_\nu(\sqrt{\lambda}\, R) = 0$.

Thus there are infinitely many eigenvalues $\lambda_{m,1}, \lambda_{m,2}, \dots,$
where $\sqrt{\lambda_{m,n}}\, R = j_{\nu,n}$, i.e., $\lambda_{m,n} = (j_{\nu,n}/R)^2$.
Associated eigenfunctions: $f_{m,n}(r) = J_\nu(j_{\nu,n}\, r/R)$.
Note that $\nu = m\pi/L$.

Eigenvalue problem:

$$\nabla^2\phi + \lambda\phi = 0 \ \text{ in } \ D = \{(r,\theta) : r < R,\ 0 < \theta < L\},$$

$$\phi|_{\partial D} = 0.$$

Eigenvalues: $\lambda_{m,n} = (j_{\frac{m\pi}{L},n}/R)^2$, where $m = 1, 2, \ldots,\ n = 1, 2, \ldots,$ and $j_{\frac{m\pi}{L},n}$ is the nth positive zero of the Bessel function $J_{\frac{m\pi}{L}}$.

Eigenfunctions:

$$\phi_{m,n}(r,\theta) = J_{\frac{m\pi}{L}}\big(j_{\frac{m\pi}{L},n} \cdot r/R\big)\sin\frac{m\pi\theta}{L}.$$