MATH 311

Topics in Applied Mathematics

Lecture 26:

Review for the final exam.

Topics for the final exam: Part I

Elementary linear algebra (Leon 1.1–1.4, 2.1–2.2)

• Systems of linear equations: elementary operations, Gaussian elimination, back substitution.

• Matrix of coefficients and augmented matrix. Elementary row operations, row echelon form and reduced row echelon form.

• Matrix algebra. Inverse matrix.

• Determinants: explicit formulas for 2×2 and 3×3 matrices, row and column expansions, elementary row and column operations.

Topics for the final exam: Part II

Abstract linear algebra (Leon 3.1–3.6, 4.1–4.3)

- Vector spaces (vectors, matrices, polynomials, functional spaces).
- Subspaces. Nullspace, column space, and row space of a matrix.
- Span, spanning set. Linear independence.
- Bases and dimension.
- Rank and nullity of a matrix.
- Coordinates relative to a basis.
- Change of basis, transition matrix.
- Linear transformations.
- Matrix transformations.
- Matrix of a linear mapping.

Topics for the final exam: Part III

Advanced linear algebra (Leon 5.1–5.6, 6.1, 6.3)

- Eigenvalues, eigenvectors, eigenspaces
- Characteristic polynomial
- Diagonalization

- Inner products and norms
- Orthogonal complement
- Least squares problems
- The Gram-Schmidt orthogonalization process

Topics for the final exam: Part IV

Applied linear algebra (Spiegel, Ch. 1, 2, 6)

- Boundary value problems
- Separation of variables
- Heat equation
- Fourier series
- Bessel functions

Problem Let V be the vector space spanned by functions $f_1(x) = x \sin x$, $f_2(x) = x \cos x$, $f_3(x) = \sin x$, and $f_4(x) = \cos x$.
Consider the linear operator $D : V \to V$, $D = d/dx$.

(a) Find the matrix A of the operator D relative to the basis f_1, f_2, f_3, f_4.
(b) Find the eigenvalues of A.
(c) Is the matrix A diagonalizable in $\mathbb{R}^4$ (in $\mathbb{C}^4$)?

A is a 4×4 matrix whose columns are coordinates of functions $Df_i = f_i'$ relative to the basis f_1, f_2, f_3, f_4.

$$f_1'(x) = (x \sin x)' = x \cos x + \sin x = f_2(x) + f_3(x),$$

$$f_2'(x) = (x \cos x)' = -x \sin x + \cos x$$
$$= -f_1(x) + f_4(x),$$

$$f_3'(x) = (\sin x)' = \cos x = f_4(x),$$

$$f_4'(x) = (\cos x)' = -\sin x = -f_3(x).$$

Thus $A = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

Eigenvalues of A are roots of its characteristic polynomial

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 1 & 0 & -\lambda & -1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix}$$

Expand the determinant by the 1st row:

$$\det(A - \lambda I) = -\lambda \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & -1 \\ 1 & 1 & -\lambda \end{vmatrix} - (-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$= \lambda^2(\lambda^2 + 1) + (\lambda^2 + 1) = (\lambda^2 + 1)^2.$$

The eigenvalues are i and $-i$, both of multiplicity 2.

Complex eigenvalues $\implies$ A is not diagonalizable in $\mathbb{R}^4$

If A is diagonalizable in $\mathbb{C}^4$ then $A = UXU^{-1}$, where U is an invertible matrix with complex entries and

$$X = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.$$

This would imply that $A^2 = UX^2U^{-1}$. But $X^2 = -I$ so that $A^2 = U(-I)U^{-1} = -I$.

$$A^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & -1 & 0 \\ 2 & 0 & 0 & -1 \end{pmatrix}.$$

Since $A^2 \neq -I$, the matrix A is not diagonalizable in $\mathbb{C}^4$.

Problem Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}$, where $\mathbf{v}_0 = (3/5, 0, -4/5)$.

(a) Find the matrix B of the operator L.
(b) Find the range and kernel of L.
(c) Find the eigenvalues of L.
(d) Find the matrix of the operator L^{311} (L applied 311 times).

$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v}, \ \ \mathbf{v}_0 = (3/5, 0, -4/5).$

Let $\mathbf{v} = (x, y, z) = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3.$ Then

$$L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 3/5 & 0 & -4/5 \\ x & y & z \end{vmatrix}$$

$$= \tfrac{4}{5}y\mathbf{e}_1 - \left(\tfrac{4}{5}x + \tfrac{3}{5}z\right)\mathbf{e}_2 + \tfrac{3}{5}y\mathbf{e}_3.$$

In particular, $L(\mathbf{e}_1) = -\tfrac{4}{5}\mathbf{e}_2,$ $L(\mathbf{e}_2) = \tfrac{4}{5}\mathbf{e}_1 + \tfrac{3}{5}\mathbf{e}_3,$
$L(\mathbf{e}_3) = -\tfrac{3}{5}\mathbf{e}_2.$

Therefore $B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$

$$B = \begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix}.$$

The range of the operator L is spanned by columns of the matrix B. It follows that $\mathrm{Range}(L)$ is the plane spanned by $\mathbf{v}_1 = (0, 1, 0)$ and $\mathbf{v}_2 = (4, 0, 3)$.

The kernel of L is the nullspace of the matrix B, i.e., the solution set for the equation $B\mathbf{x} = \mathbf{0}$.

$$\begin{pmatrix} 0 & 4/5 & 0 \\ -4/5 & 0 & -3/5 \\ 0 & 3/5 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3/4 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\implies x + \tfrac{3}{4}z = y = 0 \implies \mathbf{x} = t(-3/4, 0, 1).$$

Alternatively, the kernel of L is the set of vectors $\mathbf{v} \in \mathbb{R}^3$ such that $L(\mathbf{v}) = \mathbf{v}_0 \times \mathbf{v} = \mathbf{0}$.

It follows that this is the line spanned by $\mathbf{v}_0 = (3/5, 0, -4/5)$.

Characteristic polynomial of the matrix B:

$$\det(B - \lambda I) = \begin{vmatrix} -\lambda & 4/5 & 0 \\ -4/5 & -\lambda & -3/5 \\ 0 & 3/5 & -\lambda \end{vmatrix}$$

$$= -\lambda^3 - (3/5)^2\lambda - (4/5)^2\lambda = -\lambda^3 - \lambda = -\lambda(\lambda^2 + 1).$$

The eigenvalues are 0, i, and $-i$.

The matrix of the operator L^{311} is B^{311}.

Since the matrix B has eigenvalues 0, i, and $-i$, it is diagonalizable in $\mathbb{C}^3$. Namely, $B = UDU^{-1}$, where U is an invertible matrix with complex entries and

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}.$$

Then $B^{311} = UD^{311}U^{-1}$. We have that $D^{311} =$
$= \operatorname{diag}\big(0, i^{311}, (-i)^{311}\big) = \operatorname{diag}(0, -i, i) = -D$.
Hence

$$B^{311} = U(-D)U^{-1} = -B = \begin{pmatrix} 0 & -4/5 & 0 \\ 4/5 & 0 & 3/5 \\ 0 & -3/5 & 0 \end{pmatrix}.$$

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

The best least squares fit is a polynomial $q(x)$ that minimizes the distance relative to the integral norm

$$\|f - q\| = \left(\int_{-1}^{1} |f(x) - q(x)|^2 \, dx \right)^{1/2}$$

over all polynomials of degree 2.

The norm $\| \cdot \|$ is induced by the inner product
$$\langle g, h \rangle = \int_{-1}^{1} g(x)h(x)\,dx.$$

Therefore $\|f - p\|$ is minimal if p is the orthogonal projection of the function f on the subspace $\mathcal{P}_3$ of quadratic polynomials.

Suppose that p_0, p_1, p_2 is an orthogonal basis for $\mathcal{P}_3$. Then
$$q(x) = \frac{\langle f, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) + \frac{\langle f, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) + \frac{\langle f, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2(x).$$

An orthogonal basis can be obtained by applying the *Gram-Schmidt orthogonalization process* to the basis $1, x, x^2$:

$$p_0(x) = 1,$$

$$p_1(x) = x - \frac{\langle x, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) = x - \frac{\langle x, 1 \rangle}{\langle 1, 1 \rangle} = x,$$

$$p_2(x) = x^2 - \frac{\langle x^2, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) - \frac{\langle x^2, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x)$$

$$= x^2 - \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} - \frac{\langle x^2, x \rangle}{\langle x, x \rangle} x = x^2 - \frac{1}{3}.$$

Problem. Find a quadratic polynomial that is the best least squares fit to the function $f(x) = |x|$ on the interval $[-1, 1]$.

Solution:

$$q(x) = \frac{\langle f, p_0 \rangle}{\langle p_0, p_0 \rangle} p_0(x) + \frac{\langle f, p_1 \rangle}{\langle p_1, p_1 \rangle} p_1(x) + \frac{\langle f, p_2 \rangle}{\langle p_2, p_2 \rangle} p_2(x)$$

$$= \frac{1}{2} p_0(x) + \frac{15}{16} p_2(x)$$

$$= \frac{1}{2} + \frac{15}{16}\left(x^2 - \frac{1}{3}\right) = \frac{3}{16}(5x^2 + 1).$$