

Sample problems for Test 2

Any problem may be altered or replaced by a different one!

Problem 1 (15 pts.) Let $\mathcal{M}_{2,2}(\mathbb{R})$ denote the vector space of 2×2 matrices with real entries. Consider a linear operator $L : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R})$ given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator L with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 2 (30 pts.) Let $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$.

- (i) Find all eigenvalues of the matrix A .
- (ii) For each eigenvalue of A , find an associated eigenvector.
- (iii) Is the matrix A diagonalizable? Explain.
- (iv) Find all eigenvalues of the matrix A^2 .

Problem 3 (20 pts.) Find a linear polynomial which is the best least squares fit to the following data:

$$\begin{array}{c|c|c|c|c|c} x & -2 & -1 & 0 & 1 & 2 \\ \hline f(x) & -3 & -2 & 1 & 2 & 5 \end{array}$$

Problem 4 (25 pts.) Let V be a subspace of \mathbb{R}^4 spanned by the vectors $\mathbf{x}_1 = (1, 1, 1, 1)$ and $\mathbf{x}_2 = (1, 0, 3, 0)$.

- (i) Find an orthonormal basis for V .
- (ii) Find an orthonormal basis for the orthogonal complement V^\perp .

Bonus Problem 5 (15 pts.) Let $L : V \rightarrow W$ be a linear mapping of a finite-dimensional vector space V to a vector space W . Show that

$$\dim \text{Range}(L) + \dim \ker(L) = \dim V.$$