

## Sample problems for Test 2: Solutions

**Any problem may be altered or replaced by a different one!**

**Problem 1 (15 pts.)** Let  $\mathcal{M}_{2,2}(\mathbb{R})$  denote the vector space of  $2 \times 2$  matrices with real entries. Consider a linear operator  $L : \mathcal{M}_{2,2}(\mathbb{R}) \rightarrow \mathcal{M}_{2,2}(\mathbb{R})$  given by

$$L \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

Find the matrix of the operator  $L$  with respect to the basis

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad E_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let  $M_L$  denote the desired matrix. By definition,  $M_L$  is a  $4 \times 4$  matrix whose columns are coordinates of the matrices  $L(E_1), L(E_2), L(E_3), L(E_4)$  with respect to the basis  $E_1, E_2, E_3, E_4$ . We have that

$$L(E_1) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = 1E_1 + 2E_2 + 0E_3 + 0E_4,$$

$$L(E_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 4 \\ 0 & 0 \end{pmatrix} = 3E_1 + 4E_2 + 0E_3 + 0E_4,$$

$$L(E_3) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 2 \end{pmatrix} = 0E_1 + 0E_2 + 1E_3 + 2E_4,$$

$$L(E_4) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = 0E_1 + 0E_2 + 3E_3 + 4E_4.$$

It follows that

$$M_L = \begin{pmatrix} 1 & 3 & 0 & 0 \\ 2 & 4 & 0 & 0 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 2 & 4 \end{pmatrix}.$$

**Problem 2 (30 pts.)** Let  $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{pmatrix}$ .

(i) Find all eigenvalues of the matrix  $A$ .

The eigenvalues of  $A$  are roots of the characteristic equation  $\det(A - \lambda I) = 0$ . We obtain that

$$\det(A - \lambda I) = \begin{vmatrix} 1 - \lambda & 2 & 0 \\ 1 & 1 - \lambda & 1 \\ 0 & 2 & 1 - \lambda \end{vmatrix} = (1 - \lambda)^3 - 2(1 - \lambda) - 2(1 - \lambda)$$

$$= (1 - \lambda)((1 - \lambda)^2 - 4) = (1 - \lambda)((1 - \lambda) - 2)((1 - \lambda) + 2) = -(\lambda - 1)(\lambda + 1)(\lambda - 3).$$

Hence the matrix  $A$  has three eigenvalues:  $-1$ ,  $1$ , and  $3$ .

(ii) For each eigenvalue of  $A$ , find an associated eigenvector.

An eigenvector  $\mathbf{v} = (x, y, z)$  of  $A$  associated with an eigenvalue  $\lambda$  is a nonzero solution of the vector equation  $(A - \lambda I)\mathbf{v} = \mathbf{0}$ . To solve the equation, we apply row reduction to the matrix  $A - \lambda I$ .

First consider the case  $\lambda = -1$ . The row reduction yields

$$A + I = \begin{pmatrix} 2 & 2 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A + I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y + z = 0. \end{cases}$$

The general solution is  $x = t$ ,  $y = -t$ ,  $z = t$ , where  $t \in \mathbb{R}$ . In particular,  $\mathbf{v}_1 = (1, -1, 1)$  is an eigenvector of  $A$  associated with the eigenvalue  $-1$ .

Secondly, consider the case  $\lambda = 1$ . The row reduction yields

$$A - I = \begin{pmatrix} 0 & 2 & 0 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x + z = 0, \\ y = 0. \end{cases}$$

The general solution is  $x = -t$ ,  $y = 0$ ,  $z = t$ , where  $t \in \mathbb{R}$ . In particular,  $\mathbf{v}_2 = (-1, 0, 1)$  is an eigenvector of  $A$  associated with the eigenvalue  $1$ .

Finally, consider the case  $\lambda = 3$ . The row reduction yields

$$A - 3I = \begin{pmatrix} -2 & 2 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 1 & -2 & 1 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & 2 & -2 \end{pmatrix} \\ \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence

$$(A - 3I)\mathbf{v} = \mathbf{0} \iff \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - z = 0, \\ y - z = 0. \end{cases}$$

The general solution is  $x = t$ ,  $y = t$ ,  $z = t$ , where  $t \in \mathbb{R}$ . In particular,  $\mathbf{v}_3 = (1, 1, 1)$  is an eigenvector of  $A$  associated with the eigenvalue  $3$ .

(iii) Is the matrix  $A$  diagonalizable? Explain.

The matrix  $A$  is diagonalizable, i.e., there exists a basis for  $\mathbb{R}^3$  formed by its eigenvectors. Namely, the vectors  $\mathbf{v}_1 = (1, -1, 1)$ ,  $\mathbf{v}_2 = (-1, 0, 1)$ , and  $\mathbf{v}_3 = (1, 1, 1)$  are eigenvectors of the matrix  $A$  belonging to distinct eigenvalues. Therefore these vectors are linearly independent. It follows that  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  is a basis for  $\mathbb{R}^3$ .

Alternatively, the existence of a basis for  $\mathbb{R}^3$  consisting of eigenvectors of  $A$  already follows from the fact that the matrix  $A$  has three distinct eigenvalues.

(iv) Find all eigenvalues of the matrix  $A^2$ .

Suppose that  $\mathbf{v}$  is an eigenvector of the matrix  $A$  associated with an eigenvalue  $\lambda$ , that is,  $\mathbf{v} \neq \mathbf{0}$  and  $A\mathbf{v} = \lambda\mathbf{v}$ . Then

$$A^2\mathbf{v} = A(A\mathbf{v}) = A(\lambda\mathbf{v}) = \lambda(A\mathbf{v}) = \lambda(\lambda\mathbf{v}) = \lambda^2\mathbf{v}.$$

Therefore  $\mathbf{v}$  is also an eigenvector of the matrix  $A^2$  and the associated eigenvalue is  $\lambda^2$ . We already know that the matrix  $A$  has eigenvalues  $-1, 1,$  and  $3$ . It follows that  $A^2$  has eigenvalues  $1$  and  $9$ .

Since a  $3 \times 3$  matrix can have up to 3 eigenvalues, we need an additional argument to show that  $1$  and  $9$  are the only eigenvalues of  $A^2$ . The matrix  $A$  is diagonalizable. Namely,  $A = UBU^{-1}$ , where

$$B = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

and  $U$  is the matrix whose columns are eigenvectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ :

$$U = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Then  $A^2 = UBU^{-1}UBU^{-1} = UB^2U^{-1}$ . It follows that

$$\begin{aligned} \det(A^2 - \lambda I) &= \det(UB^2U^{-1} - \lambda I) = \det(UB^2U^{-1} - U(\lambda I)U^{-1}) \\ &= \det(U(B^2 - \lambda I)U^{-1}) = \det(U) \det(B^2 - \lambda I) \det(U^{-1}) = \det(B^2 - \lambda I). \end{aligned}$$

Thus the matrix  $A^2$  has the same characteristic polynomial as the diagonal matrix

$$B^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

Consequently, the matrices  $A^2$  and  $B^2$  have the same eigenvalues. The latter has eigenvalues  $1$  and  $9$ .

**Problem 3 (20 pts.)** Find a linear polynomial which is the best least squares fit to the following data:

$$\begin{array}{c|c|c|c|c|c} x & -2 & -1 & 0 & 1 & 2 \\ \hline f(x) & -3 & -2 & 1 & 2 & 5 \end{array}$$

We are looking for a function  $f(x) = c_1 + c_2x$ , where  $c_1, c_2$  are unknown coefficients. The data of the problem give rise to an overdetermined system of linear equations in variables  $c_1$  and  $c_2$ :

$$\begin{cases} c_1 - 2c_2 = -3, \\ c_1 - c_2 = -2, \\ c_1 = 1, \\ c_1 + c_2 = 2, \\ c_1 + 2c_2 = 5. \end{cases}$$

This system is inconsistent. We can represent it as a matrix equation  $\mathbf{A}\mathbf{c} = \mathbf{y}$ , where

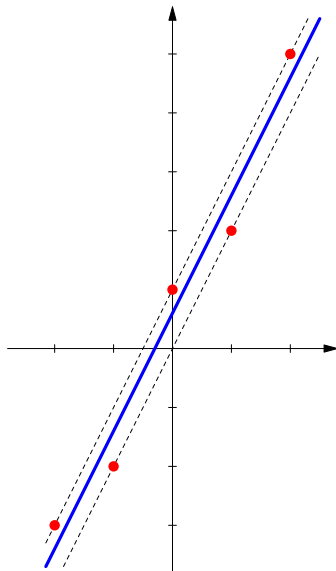
$$\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}, \quad \mathbf{c} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \quad \mathbf{y} = \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

The least squares solution  $\mathbf{c}$  of the above system is a solution of the system  $\mathbf{A}^T\mathbf{A}\mathbf{c} = \mathbf{A}^T\mathbf{y}$ :

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -3 \\ -2 \\ 1 \\ 2 \\ 5 \end{pmatrix}$$

$$\iff \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 20 \end{pmatrix} \iff \begin{cases} c_1 = 3/5 \\ c_2 = 2 \end{cases}$$

Thus the function  $f(x) = \frac{3}{5} + 2x$  is the best least squares fit to the above data among linear polynomials.



**Problem 4 (25 pts.)** Let  $V$  be a subspace of  $\mathbb{R}^4$  spanned by the vectors  $\mathbf{x}_1 = (1, 1, 1, 1)$  and  $\mathbf{x}_2 = (1, 0, 3, 0)$ .

(i) Find an orthonormal basis for  $V$ .

First we apply the Gram-Schmidt orthogonalization process to vectors  $\mathbf{x}_1, \mathbf{x}_2$  and obtain an orthogonal basis  $\mathbf{v}_1, \mathbf{v}_2$  for the subspace  $V$ :

$$\mathbf{v}_1 = \mathbf{x}_1 = (1, 1, 1, 1), \quad \mathbf{v}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (1, 0, 3, 0) - \frac{4}{4}(1, 1, 1, 1) = (0, -1, 2, -1).$$

Then we normalize vectors  $\mathbf{v}_1, \mathbf{v}_2$  to obtain an orthonormal basis  $\mathbf{w}_1, \mathbf{w}_2$  for  $V$ :

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}\mathbf{v}_1 = \frac{1}{2}(1, 1, 1, 1), \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}\mathbf{v}_2 = \frac{1}{\sqrt{6}}(0, -1, 2, -1).$$

(ii) Find an orthonormal basis for the orthogonal complement  $V^\perp$ .

Since the subspace  $V$  is spanned by vectors  $(1, 1, 1, 1)$  and  $(1, 0, 3, 0)$ , it is the row space of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix}.$$

Then the orthogonal complement  $V^\perp$  is the nullspace of  $A$ . To find the nullspace, we convert the matrix  $A$  to reduced row echelon form:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 3 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -2 & 1 \end{pmatrix}.$$

Hence a vector  $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$  belongs to  $V^\perp$  if and only if

$$\begin{cases} x_1 + 3x_3 = 0 \\ x_2 - 2x_3 + x_4 = 0 \end{cases} \iff \begin{cases} x_1 = -3x_3 \\ x_2 = 2x_3 - x_4 \end{cases}$$

The general solution of the system is  $(x_1, x_2, x_3, x_4) = (-3t, 2t - s, t, s) = t(-3, 2, 1, 0) + s(0, -1, 0, 1)$ , where  $t, s \in \mathbb{R}$ . It follows that  $V^\perp$  is spanned by vectors  $\mathbf{x}_3 = (0, -1, 0, 1)$  and  $\mathbf{x}_4 = (-3, 2, 1, 0)$ . It remains to orthogonalize and normalize this basis for  $V^\perp$ :

$$\mathbf{v}_3 = \mathbf{x}_3 = (0, -1, 0, 1), \quad \mathbf{v}_4 = \mathbf{x}_4 - \frac{\mathbf{x}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = (-3, 2, 1, 0) - \frac{-2}{2}(0, -1, 0, 1) = (-3, 1, 1, 1),$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{2}}(0, -1, 0, 1), \quad \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \frac{1}{2\sqrt{3}}\mathbf{v}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1).$$

Thus the vectors  $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$  and  $\mathbf{w}_4 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$  form an orthonormal basis for  $V^\perp$ .

*Alternative solution:* Suppose that an orthonormal basis  $\mathbf{w}_1, \mathbf{w}_2$  for the subspace  $V$  has been extended to an orthonormal basis  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$  for  $\mathbb{R}^4$ . Then the vectors  $\mathbf{w}_3, \mathbf{w}_4$  form an orthonormal basis for the orthogonal complement  $V^\perp$ .

We know that vectors  $\mathbf{v}_1 = (1, 1, 1, 1)$  and  $\mathbf{v}_2 = (0, -1, 2, -1)$  form an orthogonal basis for  $V$ . This basis can be extended to a basis for  $\mathbb{R}^4$  by adding two vectors from the standard basis. For example, we can add vectors  $\mathbf{e}_3 = (0, 0, 1, 0)$  and  $\mathbf{e}_4 = (0, 0, 0, 1)$ . The vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4$  do form a basis for  $\mathbb{R}^4$  since the matrix whose rows are these vectors is nonsingular:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} = -1 \neq 0.$$

To orthogonalize the basis  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{e}_3, \mathbf{e}_4$ , we apply the Gram-Schmidt process (note that the vectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are already orthogonal):

$$\mathbf{v}_3 = \mathbf{e}_3 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{e}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 = (0, 0, 1, 0) - \frac{1}{4}(1, 1, 1, 1) - \frac{2}{6}(0, -1, 2, -1) = \frac{1}{12}(-3, 1, 1, 1),$$

$$\begin{aligned} \mathbf{v}_4 &= \mathbf{e}_4 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \mathbf{v}_2 - \frac{\mathbf{e}_4 \cdot \mathbf{v}_3}{\mathbf{v}_3 \cdot \mathbf{v}_3} \mathbf{v}_3 = \\ &= (0, 0, 0, 1) - \frac{1}{4}(1, 1, 1, 1) - \frac{-1}{6}(0, -1, 2, -1) - \frac{1/12}{1/12} \cdot \frac{1}{12}(-3, 1, 1, 1) = \frac{1}{2}(0, -1, 0, 1). \end{aligned}$$

It remains to normalize vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ :

$$\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \frac{1}{2}(1, 1, 1, 1), \quad \mathbf{w}_2 = \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{6}}(0, -1, 2, -1),$$

$$\mathbf{w}_3 = \frac{\mathbf{v}_3}{\|\mathbf{v}_3\|} = \sqrt{12} \mathbf{v}_3 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1), \quad \mathbf{w}_4 = \frac{\mathbf{v}_4}{\|\mathbf{v}_4\|} = \sqrt{2} \mathbf{v}_4 = \frac{1}{\sqrt{2}}(0, -1, 0, 1).$$

We have obtained an orthonormal basis  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4$  for  $\mathbb{R}^4$  that extends an orthonormal basis  $\mathbf{w}_1, \mathbf{w}_2$  for the subspace  $V$ . It follows that  $\mathbf{w}_3 = \frac{1}{2\sqrt{3}}(-3, 1, 1, 1)$ ,  $\mathbf{w}_4 = \frac{1}{\sqrt{2}}(0, -1, 0, 1)$  is an orthonormal basis for  $V^\perp$ .

**Bonus Problem 5 (15 pts.)** Let  $L : V \rightarrow W$  be a linear mapping of a finite-dimensional vector space  $V$  to a vector space  $W$ . Show that

$$\dim \text{Range}(L) + \dim \ker(L) = \dim V.$$

The kernel  $\ker(L)$  is a subspace of  $V$ . Since the vector space  $V$  is finite-dimensional, so is  $\ker(L)$ . Take a basis  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_k$  for the subspace  $\ker(L)$ , then extend it to a basis  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_m$  for the entire space  $V$ . We are going to prove that vectors  $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$  form a basis for the range  $L(V)$ . Then  $\dim \text{Range}(L) = m$ ,  $\dim \ker(L) = k$ , and  $\dim V = k + m$ .

*Spanning:* Any vector  $\mathbf{w} \in \text{Range}(L)$  is represented as  $\mathbf{w} = L(\mathbf{v})$ , where  $\mathbf{v} \in V$ . We have

$$\mathbf{v} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \dots + \alpha_k \mathbf{v}_k + \beta_1 \mathbf{u}_1 + \beta_2 \mathbf{u}_2 + \dots + \beta_m \mathbf{u}_m$$

for some  $\alpha_i, \beta_j \in \mathbb{R}$ . It follows that

$$\mathbf{w} = L(\mathbf{v}) = \alpha_1 L(\mathbf{v}_1) + \dots + \alpha_k L(\mathbf{v}_k) + \beta_1 L(\mathbf{u}_1) + \dots + \beta_m L(\mathbf{u}_m) = \beta_1 L(\mathbf{u}_1) + \dots + \beta_m L(\mathbf{u}_m)$$

( $L(\mathbf{v}_i) = \mathbf{0}$  since  $\mathbf{v}_i \in \ker(L)$ ). Thus  $\text{Range}(L)$  is spanned by the vectors  $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$ .

*Linear independence:* Suppose that  $t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \dots + t_m L(\mathbf{u}_m) = \mathbf{0}$  for some  $t_i \in \mathbb{R}$ . Let  $\mathbf{u} = t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_m \mathbf{u}_m$ . Since

$$L(\mathbf{u}) = t_1 L(\mathbf{u}_1) + t_2 L(\mathbf{u}_2) + \dots + t_m L(\mathbf{u}_m) = \mathbf{0},$$

the vector  $\mathbf{u}$  belongs to the kernel of  $L$ . Therefore  $\mathbf{u} = s_1 \mathbf{v}_1 + s_2 \mathbf{v}_2 + \dots + s_k \mathbf{v}_k$  for some  $s_j \in \mathbb{R}$ . It follows that

$$t_1 \mathbf{u}_1 + t_2 \mathbf{u}_2 + \dots + t_m \mathbf{u}_m - s_1 \mathbf{v}_1 - s_2 \mathbf{v}_2 - \dots - s_k \mathbf{v}_k = \mathbf{u} - \mathbf{u} = \mathbf{0}.$$

Linear independence of vectors  $\mathbf{v}_1, \dots, \mathbf{v}_k, \mathbf{u}_1, \dots, \mathbf{u}_m$  implies that  $t_1 = t_2 = \dots = t_m = 0$  (as well as  $s_1 = s_2 = \dots = s_k = 0$ ). Thus the vectors  $L(\mathbf{u}_1), L(\mathbf{u}_2), \dots, L(\mathbf{u}_m)$  are linearly independent.