MATH 311–503/505 Fall 2015

# Sample problems for the final exam

**Any problem may be altered or replaced by a different one!**

**Problem 1** Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

**Problem 2** Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 1, 1),\ \mathbf{v}_2 = (1, 2, 2).$$

(i) Find the matrix of the operator $L$.
(ii) Find the dimensions of the range and the kernel of $L$.
(iii) Find bases for the range and the kernel of $L$.

**Problem 3** Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.
(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
(ii) Find the matrix of the operator $L$ relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
(iii) Find the matrix of the operator $L$ relative to the standard basis.

**Problem 4** Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.
(i) Find all eigenvalues of the matrix $B$.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of $B$.
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of $B$.
(iv) Find a diagonal matrix $D$ and an invertible matrix $U$ such that $B = UDU^{-1}$.

**Problem 5** Let $V$ be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.
(i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace $V$.
(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

**Problem 6** Consider a vector field $\mathbf{F}(x, y, z) = xyz\mathbf{e}_1 + xy\mathbf{e}_2 + x^2\mathbf{e}_3$.
(i) Find $\text{curl}(\mathbf{F})$.
(ii) Find the integral of the vector field $\text{curl}(\mathbf{F})$ along a hemisphere $H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$. Orient the hemisphere by the normal vector $\mathbf{n} = (0, 0, 1)$ at the point $(0, 0, 1)$.

**Problem 7** Find the area of a pentagon with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, $(3, 4)$, and $(-1, 2)$.