MATH 311

Topics in Applied Mathematics I

Lecture 32:
More on the differential.
Review of integral calculus.

The differential

Suppose V and W are normed vector spaces and consider a function $F : X \to V$, where $X \subset W$.

Definition. We say that the function F is **differentiable** at a point $\mathbf{a} \in X$ if it is defined in a neighborhood of $\mathbf{a}$ and there exists a continuous linear transformation $L : W \to V$ such that

$$F(\mathbf{a} + \mathbf{v}) = F(\mathbf{a}) + L(\mathbf{v}) + R(\mathbf{v}),$$

where $\|R(\mathbf{v})\|/\|\mathbf{v}\| \to 0$ as $\|\mathbf{v}\| \to 0$. The transformation L is called the **differential** of F at $\mathbf{a}$ and denoted $(DF)(\mathbf{a})$.

Remarks. • A linear transformation $L : W \to V$ is continuous if and only if $\|L(\mathbf{v})\| \leq C\|\mathbf{v}\|$ for some $C > 0$ and all $\mathbf{v} \in W$.

• If $\dim W < \infty$ then any linear transformation $L : W \to V$ is continuous. Otherwise it is not so.

Examples

- Any linear transformation $L : \mathbb{R} \to \mathbb{R}$ is a scaling $L(x) = rx$ by a scalar r. If L is the differential of a function $f : X \to \mathbb{R}$ at a point $a \in \mathbb{R}$, then $r = f'(a)$.

- Any linear transformation $L : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation: $L(\mathbf{x}) = B\mathbf{x}$, where $B = (b_{ij})$ is an $m \times n$ matrix. If L is the differential of a function $\mathbf{F} : X \to \mathbb{R}^m$ at a point $\mathbf{a} \in \mathbb{R}^n$, then $b_{ij} = \dfrac{\partial F_i}{\partial x_j}(\mathbf{a})$.

The matrix B of partial derivatives is called the **Jacobian matrix** of $\mathbf{F}$ and denoted $\dfrac{\partial(F_1, \dots, F_m)}{\partial(x_1, \dots, x_n)}$.

Riemann sums and Riemann integral

Definition. A **Riemann sum** of a function $f : [a, b] \to \mathbb{R}$ with respect to a partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ generated by samples $t_j \in [x_{j-1}, x_j]$ is a sum

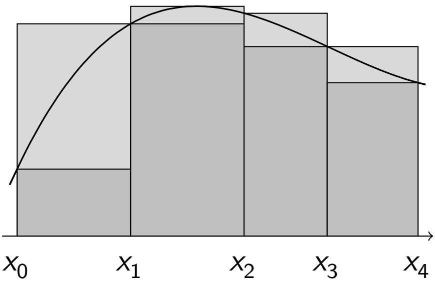
$$\mathcal{S}(f, P, t_j) = \sum_{j=1}^{n} f(t_j)\,(x_j - x_{j-1}).$$

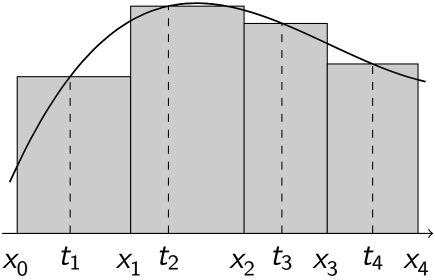
Remark. $P = \{x_0, x_1, \ldots, x_n\}$ is a partition of $[a, b]$ if $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$. The norm of the partition P is $\|P\| = \max_{1 \le j \le n} |x_j - x_{j-1}|$.

Definition. The Riemann sums $\mathcal{S}(f, P, t_j)$ **converge** to a limit $I(f)$ as the norm $\|P\| \to 0$ if for every $\varepsilon > 0$ there exists $\delta > 0$ such that $\|P\| < \delta$ implies $|\mathcal{S}(f, P, t_j) - I(f)| < \varepsilon$ for any partition P and choice of samples t_j.

If this is the case, then the function f is called **integrable** on $[a, b]$ and the limit $I(f)$ is called the **integral** of f over $[a, b]$, denoted $\int_a^b f(x)\,dx$.

Riemann sums and Darboux sums





Integration as a linear operation

Theorem 1 If functions f, g are integrable on an interval $[a, b]$, then the sum $f + g$ is also integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Theorem 2 If a function f is integrable on $[a, b]$, then for each $\alpha \in \mathbb{R}$ the scalar multiple αf is also integrable on $[a, b]$ and

$$\int_a^b \alpha f(x)\,dx = \alpha \int_a^b f(x)\,dx.$$

More properties of integrals

Theorem If a function f is integrable on $[a, b]$ and $f([a, b]) \subset [A, B]$, then for each continuous function $g : [A, B] \to \mathbb{R}$ the composition $g \circ f$ is also integrable on $[a, b]$.

Theorem If functions f and g are integrable on $[a, b]$, then so is fg.

Theorem If a function f is integrable on $[a, b]$, then it is integrable on each subinterval $[c, d] \subset [a, b]$. Moreover, for any $c \in (a, b)$ we have

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Comparison theorems for integrals

Theorem 1 If functions f, g are integrable on $[a, b]$ and $f(x) \le g(x)$ for all $x \in [a, b]$, then

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Theorem 2 If f is integrable on $[a, b]$ and $f(x) \ge 0$ for $x \in [a, b]$, then $\displaystyle\int_a^b f(x)\,dx \ge 0$.

Theorem 3 If f is integrable on $[a, b]$, then the function $|f|$ is also integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx.$$

Fundamental theorem of calculus

Theorem If a function f is continuous on an interval $[a, b]$, then the function

$$F(x) = \int_a^x f(t)\,dt, \quad x \in [a, b],$$

is continuously differentiable on $[a, b]$. Moreover, $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem If a function F is differentiable on $[a, b]$ and the derivative F' is integrable on $[a, b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Change of the variable in an integral

Theorem If ϕ is continuously differentiable on a closed, nondegenerate interval $[a, b]$ and f is continuous on $\phi([a, b])$, then

$$\int_{\phi(a)}^{\phi(b)} f(t)\, dt = \int_a^b f(\phi(x))\, \phi'(x)\, dx = \int_a^b f(\phi(x))\, d\phi(x).$$

Remarks. • It is possible that $\phi(a) \geq \phi(b)$. To make sense of the integral in this case, we set

$$\int_c^d f(t)\, dt = -\int_d^c f(t)\, dt$$

if $c > d$. Also, we set the integral to be 0 if $c = d$.

• $t = \phi(x)$ is a proper change of the variable only if the function ϕ is strictly monotone. However the theorem holds even without this assumption.

Sets of measure zero

Definition. A subset E of the real line $\mathbb{R}$ is said to have **measure zero** if for any $\varepsilon > 0$ the set E can be covered by a sequence of open intervals $J_1, J_2, \dots$ such that $\sum_{n=1}^{\infty} |J_n| < \varepsilon$.

Examples. • Any set E that can be represented as a sequence $x_1, x_2, \dots$ (such sets are called **countable**) has measure zero.
Indeed, for any $\varepsilon > 0$, let

$$J_n = \left(x_n - \frac{\varepsilon}{2^{n+1}}, x_n + \frac{\varepsilon}{2^{n+1}} \right), \quad n = 1, 2, \dots$$

Then $E \subset J_1 \cup J_2 \cup \dots$ and $|J_n| = \varepsilon / 2^n$ for all $n \in \mathbb{N}$ so that $\sum_{n=1}^{\infty} |J_n| = \varepsilon$.

• The set $\mathbb{Q}$ of rational numbers has measure zero (since it is countable).

• Nondegenerate interval $[a, b]$ is not a set of measure zero.

Lebesgue's criterion for Riemann integrability

Definition. Suppose $P(x)$ is a property depending on $x \in S$, where $S \subset \mathbb{R}$. We say that $P(x)$ holds for **almost all** $x \in S$ (or **almost everywhere** on S) if the set $\{x \in S \mid P(x) \text{ does not hold}\}$ has measure zero.

Theorem A function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on the interval $[a, b]$ if and only if f is bounded on $[a, b]$ and continuous almost everywhere on $[a, b]$.

Area, volume, and determinants

- 2×2 determinants and plane geometry

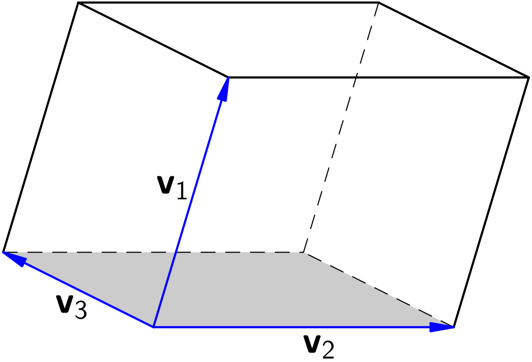
Let P be a parallelogram in the plane $\mathbb{R}^2$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^2$ are represented by adjacent sides of P. Then $\mathrm{area}(P) = |\det A|$, where $A = (\mathbf{v}_1, \mathbf{v}_2)$, a matrix whose columns are $\mathbf{v}_1$ and $\mathbf{v}_2$.

Consider a linear operator $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ given by $L_A(\mathbf{v}) = A\mathbf{v}$ for any column vector $\mathbf{v}$. Then $\mathrm{area}(L_A(D)) = |\det A|\,\mathrm{area}(D)$ for any bounded domain D.

- 3×3 determinants and space geometry

Let Π be a parallelepiped in space $\mathbb{R}^3$. Suppose that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^3$ are represented by adjacent edges of Π. Then $\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, a matrix whose columns are $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Similarly, $\mathrm{volume}(L_B(D)) = |\det B|\,\mathrm{volume}(D)$ for any bounded domain $D \subset \mathbb{R}^3$.



$\mathrm{volume}(\Pi) = |\det B|$, where $B = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$. Note that the parallelepiped Π is the image under L_B of a unit cube whose adjacent edges are $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.

The triple $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ obeys the right-hand rule. We say that L_B **preserves orientation** if it preserves the hand rule for any basis. This is the case if and only if $\det B > 0$.