MATH 311–503/505 Fall 2016

Sample problems for the final exam

Any problem may be altered or replaced by a different one!

Problem 1 Find the point of intersection of the planes $x + 2y - z = 1$, $x - 3y = -5$, and $2x + y + z = 0$ in $\mathbb{R}^3$.

Problem 2 Consider a linear operator $L : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$L(\mathbf{v}) = (\mathbf{v} \cdot \mathbf{v}_1)\mathbf{v}_2, \quad \text{where } \mathbf{v}_1 = (1, 1, 1),\ \mathbf{v}_2 = (1, 2, 2).$$

(i) Find the matrix of the operator L.
(ii) Find the dimensions of the range and the kernel of L.
(iii) Find bases for the range and the kernel of L.

Problem 3 Let $\mathbf{v}_1 = (1, 1, 1)$, $\mathbf{v}_2 = (1, 1, 0)$, and $\mathbf{v}_3 = (1, 0, 1)$. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator on $\mathbb{R}^3$ such that $L(\mathbf{v}_1) = \mathbf{v}_2$, $L(\mathbf{v}_2) = \mathbf{v}_3$, $L(\mathbf{v}_3) = \mathbf{v}_1$.
(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ form a basis for $\mathbb{R}^3$.
(ii) Find the matrix of the operator L relative to the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.
(iii) Find the matrix of the operator L relative to the standard basis.

Problem 4 Let $B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$.

(i) Find all eigenvalues of the matrix B.
(ii) Find a basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iii) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors of B.
(iv) Find a diagonal matrix D and an invertible matrix U such that $B = UDU^{-1}$.

Problem 5 Let V be a subspace of $\mathbb{R}^4$ spanned by vectors $\mathbf{x}_1 = (1, 1, 0, 0)$, $\mathbf{x}_2 = (2, 0, -1, 1)$, and $\mathbf{x}_3 = (0, 1, 1, 0)$.
(i) Find the distance from the point $\mathbf{y} = (0, 0, 0, 4)$ to the subspace V.
(ii) Find the distance from the point $\mathbf{y}$ to the orthogonal complement $V^\perp$.

Problem 6 Consider a vector field $\mathbf{F}(x, y, z) = xyz\mathbf{e}_1 + xy\mathbf{e}_2 + x^2\mathbf{e}_3$.
(i) Find $\text{curl}(\mathbf{F})$.
(ii) Find the integral of the vector field $\text{curl}(\mathbf{F})$ along a hemisphere $H = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$. Orient the hemisphere by the normal vector $\mathbf{n} = (0, 0, 1)$ at the point $(0, 0, 1)$.

Problem 7 Find the area of a pentagon with vertices $(0, 0)$, $(4, 0)$, $(5, 2)$, $(3, 4)$, and $(-1, 2)$.